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THE b-CHROMATIC NUMBER OF HELM GRAPH

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Abstract: The b-chromatic number $\varphi(G)$ of a graph G is the greatest integer k such that G admits a proper k-coloring in which every color class i has a vertex realizing color i that is proficient to correspond with all the others color classes. The paper estimates the b-chromatic number of helm graph, central graph of helm graph and middle graph of helm graph which is denoted by H_n, $C(H_n)$ and $M(H_n)$ respectively.

Keywords: b-chromatic number; Helm graph; Central graph; Middle graph
2010 AMS MSC: 05C15, 05C76

1. INTRODUCTION AND PRELIMINARIES

All graphs in this note are finite and simple. For notation not defined here we refer to Harary [5]. A vertex coloring of a graph G is a function that maps vertices of graph to a set of positive integer. If the adjacent vertices receive the different colors then minimum number of colors k is the chromatic number of graph G and it is known as proper k-coloring. The area of graph coloring is quite old and still very active. Many problems can be formulated as a graph coloring problem including channel assignment in radio stations, time tabling, clustering in data mining, scheduling, automatic reading system, micro-economics and register allocation etc. There are many problems in graph coloring, one such problem is b-coloring problem. We refer the color class as a community. The b-coloring of graph G is a proper k-coloring in which every communities contains a vertex that is adjacent to at least one vertex in each of other communities. Such a vertex is called a color-dominating vertex. We denote by $\varphi(G)$ the maximum number k for which there exists a b-coloring of graph G using k colors. This parameter of b-coloring was introduced by Irving and Manlove [12], and is called the b-chromatic number of graph G. In that paper they prove that determining the b-chromatic number is NP-hard in general and polynomial for trees. Let $m(G)$ be the largest integer k such that G has k vertices of degree at least $k-1$, and let $\Delta(G)$ be the maximum degree in G. For a given graph G, it may be easily remarked that $b(G) \leq m(G) \leq \Delta(G)+1$.
This parameter has received exceptional consideration by many authors. Kratochvl et al. [11] showed that determining $\varphi(G)$ is NP-hard even for bipartite graphs. Corteel et al. [4] proved that there is no constant $\epsilon > 0$ for which this problem can be approximated within a factor of $120/113-\epsilon$ in polynomial time, unless $P = NP$. Hoang and Kouider [6] characterize all bipartite graphs G and all P4-sparse graphs G such that each induced subgraph H of G satisfies $b(H) = \chi(H)$, where $\chi(H)$ is the chromatic number of H. The b-chromatic number of the Cartesian product of general graphs was studied by Kouider and Zaker [9] and Kouider and Maheo [10]. Exact value of the b-chromatic number of central graph, middle graph, total graph and line graph of star graphs has been premeditated by Vijayalaksmi et al. [13]. Vivin and Venkatachalam [14] investigated the b-chromatic number of corona graph of any graph with path, cycle and complete graph.
For some special families of graphs, some authors have obtained upper or lower bound for $\varphi(G)$. The bounds for the b-chromatic number have been studied by Kouider and Maheo [7] in general and for some graph classes, see Balakrishnan et al. [2], and Chaouche and Berrachedi [3]. We also recall the following result of Kouider [8]. Let G be a graph of girth at least 5 then $\varphi(G) > min\left\{\delta, \frac{D}{6}\right\}$, for minimum degree δ and diameter D.

Let G be a graph with vertex set $V(G)$ and the edge set $E(G)$. The central graph [1] of G, denoted $C(G)$ is obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices of G.

The middle graph [15] of G denoted by $M(G)$, is defined with the vertex set $V(G) \cup E(G)$ where two vertices are adjacent iff they are either adjacent edges of G or one is a vertex and the other is an edge incident with it.

In this paper, we study the b-coloring of helm graph H_n, central graph of helm graph $C(H_n)$, and middle graph of helm graph $M(H_n)$ and we obtain the b-chromatic number for these graphs.

2. HELM GRAPH

A Helm graph H_n of order n which contains a cycle of order $n-1$, for which every graph vertex in the cycle is connected to hub and each vertex of the cycle adjoin by a pendant edge.

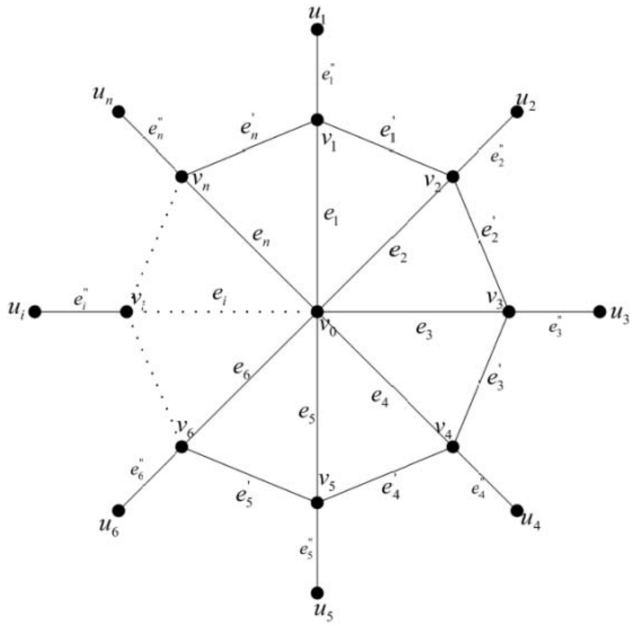

Figure 1 Helm graph H_n.

Theorem 2.1.

If $n \geq 6, \varphi\{H_n\} = 5$, n is the number of vertices in H_n.

Proof.

Let H_n be a helm graph. Let $V\{H_n\} = \{v_0\} \cup \{v_i: 1 \leq i \leq n\} \cup \{u_i: 1 \leq i \leq n\}$ (taken in order clockwise); where v_0 is the hub, v_i for $1 \leq i \leq n$ are the vertex set of cycle and u_i for $1 \leq i \leq n$ are the pendant vertices such that each $v_i u_i$ is a pendant edge. Clearly $\deg(v_0) = n$ and $\deg(v_i) = 4$ for $1 \leq i \leq n$. We assign 5 colors, $C = \{1,2,3,4,5\}$ to the vertices of H_n as follows: Assign i to v_i for $1 \leq i \leq 4$, 4 to v_n, 5 to v_0, 3 to u_1, 4 to u_2, 1 to u_3 and u_4, 5 to u_i for $5 \leq i \leq n$. Now assign 2 to v_{2i-1} for $n \geq 6$ and $3 \leq i \leq \left\lfloor\frac{n}{2}\right\rfloor$; and 3 to v_{2i} for $n \geq 7$ and $3 \leq i \leq \left\lfloor\frac{n-1}{2}\right\rfloor$. First claim that above said coloring is proper coloirng since no two adjacent vertices of H_n receive same color, therefore it is proper coloring. Second claim that this coloring is b-chromatic since each color class contains a vertex that has an adjacent vertices in all other color class and it is maximal since $\varphi\{H_n\} = m(G) = 5$, so that above said coloring is b-chromatic and has size 5. Hence $\varphi\{H_n\} = 5$, for $n \geq 6$ (see Fig. 1). {Note: $\varphi\{H_3\} = \varphi\{H_5\} = 4$ and $\varphi\{H_4\} = 5$}. □

Theorem 2.2.

If $n \geq 3$, $\varphi\{C(H_n)\} = n + 1 + \left\lfloor\frac{n}{2}\right\rfloor$, n is the number of vertices in H_n.

Proof.

Let H_n be a helm graph. Let $V\{H_n\} = \{v_0\} \cup \{v_i: 1 \leq i \leq n\} \cup \{u_i: 1 \leq i \leq n\}$ (taken in order clockwise); where v_0 is the hub, v_i for $1 \leq i \leq n$ are the vertex set of cycle and u_i for $1 \leq i \leq n$ are the pendant vertices. Now let $E(H_n) = \{e_i: 1 \leq i \leq n\} \cup \{e_i': 1 \leq i \leq n-1\} \cup \{e_i'': 1 \leq i \leq n\} \cup \{e_n'\}$; where e_i is the edge $v_0 v_i$ (for $1 \leq i \leq n$), e_i' is the edge $v_i v_{i+1}$ (for $1 \leq i \leq n-1$), e_i'' is the edge $v_i u_i$ (for $1 \leq i \leq n$) and e_n' is edge $v_n v_1$.
By definition of central graph each edge of a graph is subdivided by new vertex. Let us assume that each edge e_i, e_i', e_i'' is subdivided by v_i'', v_i' and u_i' for $1 \leq i \leq n$ respectively and joining all non adjacent vertices. So that $V(C(H_n)) = V(Y_n) \cup E(Y_n) = \{v_0\} \cup \{v_i: 1 \leq i \leq n\} \cup \{u_i: 1 \leq i \leq n\} \cup \{v_i'': 1 \leq i \leq n\} \cup \{v_i': 1 \leq i \leq n\} \cup \{u_i': 1 \leq i \leq n\}$.
A procedure to obtain b-chromatic number of $C(H_n)$ is as follows:
First we find $t = \left\lfloor\frac{n}{2}\right\rfloor$, where n is number of vertices in H_n. Assign the color $i+1$ to v_i'' for $1 \leq i \leq n-1$, 1 to v_n'' and u_n', $i+t$ to v_i' and u_i for $1 \leq i \leq n$, $n+1+t$ to v_0 and u_i' for $1 \leq i \leq n-1$. If n is odd then assign $n+1+t$ to v_n. Now assign i to v_{2i-1} and v_{2i}; for $1 \leq i \leq t$.
If $\varphi\{C(H_n)\} = n + \left\lfloor\frac{n}{2}\right\rfloor + 2$ for $n \geq 3$ then by upper bound given Irving and Manlove [12], there must be at least $n + \left\lfloor\frac{n}{2}\right\rfloor + 2$ vertices of degree $n + \left\lfloor\frac{n}{2}\right\rfloor + 1$ in $C(H_n)$, all with distinct colors and each neighbours to vertices of the other color. E.g. if we assign distinct colors a and b to v_1 and v_2 respectively then pendant vertices u_1 and u_2 can't have adjacent vertices of color a and b respectively, so that any recipe of colors fails to lift up new color. Therefore $n + \left\lfloor\frac{n}{2}\right\rfloor + 2$ coloring is not possible. Thus $\varphi\{C(H_n)\} \leq n + \left\lfloor\frac{n}{2}\right\rfloor + 1$. Hence $\varphi\{C(H_n)\} = n + \left\lfloor\frac{n}{2}\right\rfloor + 1$, for $n \geq 3$ (see Fig. 2). □

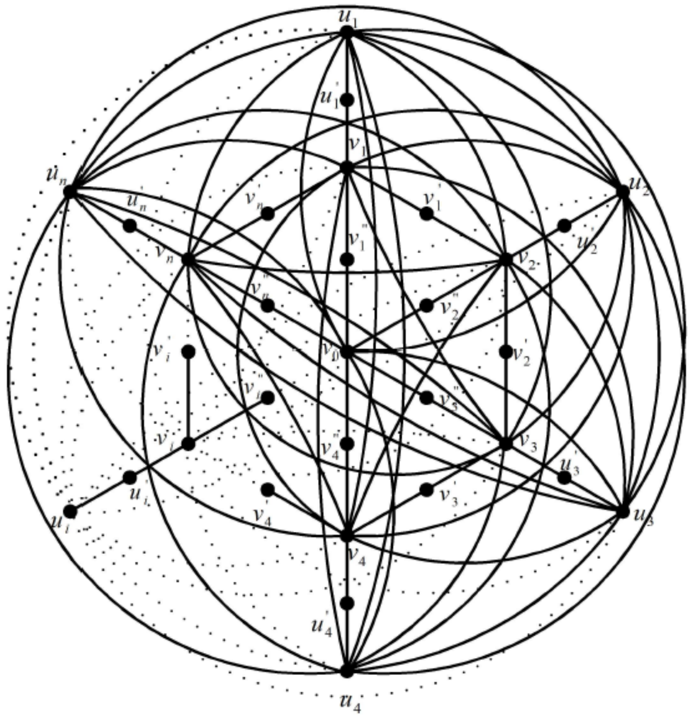

Figure 2 Central graph of Helm graph $C(H_n)$.

Theorem 2.3.

If $n \geq 8$, $\varphi\{M(H_n)\} = n + 1$, n is the number of vertices in H_n.

Proof.

Let H_n be a helm graph of order n. let $V\{H_n\} = \{v_0\} \cup \{v_i: 1 \leq i \leq n\} \cup \{u_i: 1 \leq i \leq n\}$ (taken in order clockwise); where v_0 is the hub, v_i for $1 \leq i \leq n$ are the vertex set of cycle and u_i for $1 \leq i \leq n$ are the pendant vertices. Now let $E(H_n) = \{e_i: 1 \leq i \leq n\} \cup \{e_i': 1 \leq i \leq n-1\} \cup \{e_i'': 1 \leq i \leq n\} \cup \{e_n'\}$; where e_i is the edge v_0v_i (for $1 \leq i \leq n$), e_i' is the edge v_iv_{i+1} (for $1 \leq i \leq n-1$), e_i'' is the edge v_iu_i (for $1 \leq i \leq n$) and e_n' is edge v_nv_1.
By definition of middle graph each edge of helm graph is subdivided by new vertex therefore assume that each edge e_i, e_i' and e_i'' is subdivided by v_i'', v_i' and u_i' for $1 \leq i \leq n$ respectively. So that $V(C(H_n)) = V(Y_n) \cup E(Y_n) = \{v_0\} \cup \{v_i: 1 \leq i \leq n\} \cup \{u_i: 1 \leq i \leq n\} \cup \{v_i'': 1 \leq i \leq n\} \cup \{v_i': 1 \leq i \leq n\} \cup \{u_i': 1 \leq i \leq n\}$. Note that the vertices v_i'', $1 \leq i \leq n$ with v_0 induces a clique of order $n+1$ in $M(H_n)$, i.e. $\varphi\{M(H_n)\} \geq n + 1$ (where n is the number of vertices in H_n). Assign proper coloring to the vertices of $M(H_n)$ as follows:

Assign $i + 2$ to v_i for $1 \leq i \leq 5$, 2 to v_i for $6 \leq i \leq n$, 7 to v_1', 8 to v_2', 9 to v_3', 8 to v_4', 4 to v_5', 8 to v_6'. Assign 5 to v_{2k-1}' for $n \geq 8$ and $4 \leq k \leq \left\lceil \frac{n}{2} \right\rceil$ and 4 to v_{2k}' for $n \geq 8$ and $4 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$.
Now assign 8 to u_1', 6 to u_2', 1 to u_i' for $3 \leq i \leq n$, 3 to u_i for $1 \leq i \leq n$. Now assign i to v_i'' for $1 \leq i \leq n$ and $n+1$ to v_0. We get a b-coloring with b-vertices v_i'' (for $1 \leq i \leq n$) and v_0 for the color classes $1, 2, 3, \ldots, n$ and $n+1$ respectively. It is maximal since $\varphi\{M(H_n)\} = m(G)$. Hence $\varphi\{M(H_n)\} = n + 1$, $n \geq 8$. (see Fig. 3). □

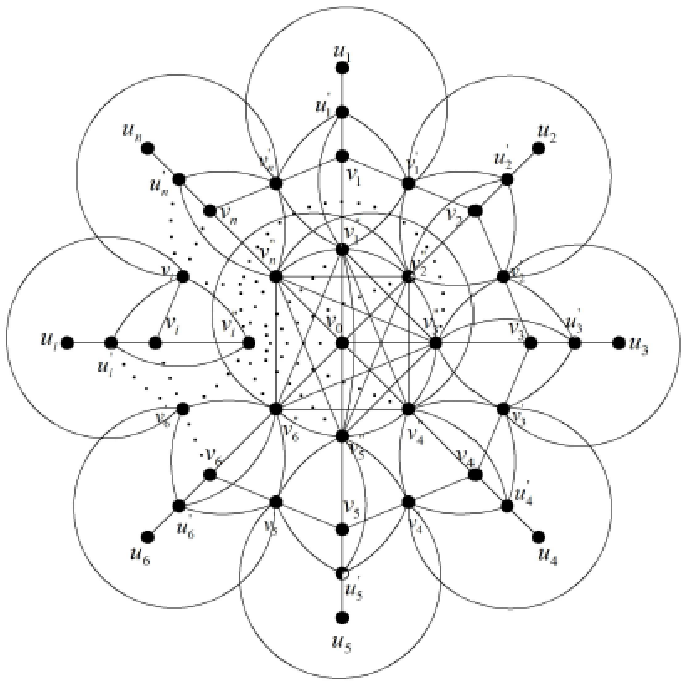

Figure 3 Middle graph of Helm graph $M(H_n)$.

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