Volume 8, No. 6, July 2017 (Special Issue III)

ISSN No. 0976-5697

International Journal of Advanced Research in Computer Science

RESEARCH PAPER

Available Online at www.ijarcs.info

INDEPENDENT DOMINATIONS IN DIRECT PRODUCT GRAPHS ARISING FROM EULER TOTIENT CAYLEY GRAPHS AND ARITHMETIC GRAPHS

S.Uma Maheswari
epartment of Mathematics,
J.M.J.College for Women,
Tenali,Andhra Pradesh,India.

B.Maheswari
Department of Applied Mathematics
S.P. Women's University,
Tirupati, Andhra Pradesh.

ABSTRACTGraph Theory is one of the most flourishing branches of modern Mathematics finding widest applications in all most all branches of Science & Technology. It is applied in diverse areas such as social sciences, linguistics, physical sciences, communication engineering etc. Number Theory is one of the oldest branches of Mathematics, which inherited rich contributions from almost all greatest mathematicians, ancient and modern. Every branch of Mathematics employs some notion of a product that enables the combination or decomposition of its elemental structures. Product of graphs are introduced in graph theory very recently and developing rapidly. In this paper, we consider direct product graphs of Cayley graphs with Arithmetic graphs and present independent dominating set of these graphs.

Keywords: Euler totient Cayley graph, Arithmetic graph, direct product graph, dominating set and independent dominating set.
AMS (MOS) Subject Classification: 6905c

1. INTRODUCTION

'Domination in graphs' is the fast growing area in Graph Theory that has emerged rapidly in the last four decades. Domination in graphs has applications to several fields such as facility location problems, School Bus Routing, Computer Communication Networks, Radio Stations, Locating Radar Stations Problem etc., Number Theory is one of the oldest branches of mathematics, which inherited rich contributions from almost all great mathematicians, ancient and modern. Nathanson [1] was the pioneer in introducing the concepts of Number Theory, particularly, the „Theory of Congruences‟ in Graph Theory, and paved the way for the emergence of a new class of graphs, namely "Arithmetic Graphs". Cayley Graphs are another class of graphs associated with elements of a group. If this group is associated with some Arithmetic function then the Cayley graph becomes an Arithmetic graph. The Cayley graph associated with Euler totient function is called an Euler totient Cayley graph. Products are often viewed as a convenient language with which one can describe structures, but they are increasingly being applied in more substantial ways. Computer Science is one of the many fields in which graph products are becoming common place.

Now we present necessary definitions, observations and some useful results that we need for next sections.

Dominating set

A subset D of $V(G)$ is said to be a dominating set of G if every vertex in $V - D$ is adjacent to a vertex in D.

The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$.

Independent dominating set

A dominating set D in which no two vertices are adjacent is called an independent dominating set of G.

The induced subgraph $\langle D \rangle$ is a null graph if D is an independent dominating set.

The minimum cardinality of an independent dominating set of G is called the independent domination number of G and is denoted by $\gamma_i(G)$.

Euler Totient Cayley Graph $G(Z_n, \varphi)$ and its Properties

Madhavi [2] introduced the concept of Euler totient Cayley graphs and studied some of its properties. She gave methods of enumeration of disjoint Hamilton cycles and triangles in these graphs.

For any positive integer n, let $Z_n = \{0,1,2, \ldots\ldots n-1\}$. Then $(Z_n, \oplus)$, where, $\oplus$ is addition modulo n, is an abelian group of order n. The number of positive integers less than n and relatively prime to n is denoted by $\varphi(n)$ and is called Euler totient function.

Let S denote the set of all positive integers less than n and relatively prime to n .That is $S = \{r / 1 \leq r < n$ and GCD $r, n = 1$. Then $S = \varphi n$.

Now we define Euler totient Cayley graph as follows.

For each positive integer n , let Z_n be the additive group of integers modulo n and let S be the set of all integers less than n and relatively prime to n. The Euler totient Cayley graph $G(Z_n, \varphi)$ is defined as the graph whose vertex set V is given by $Z_n = \{0,1,2, \ldots . n-1\}$ and the edge set is $E = \{(x, y) / x - y \in S$ or $y - x \in S\}$.

Clearly as proved in [2], the Euler totient Cayley graph $G(Z_n, \varphi)$ is

1. a connected, simple and undirected graph,

CONFERENCE PAPER
National Conference dated 27-28 July 2017 on
Recent Advances in Graph Theory and its Applications (NCRAGTA2017)
Organized by Dept of Applied Mathematics
Sri Padmawati Mahila Vishvavidyalayam (Women's University) Tirupati, A.P.,

2. $\varphi(n)$ - regular and has $\frac{n.\varphi(n)}{2}$ edges,
3. Hamiltonian,
4. Eulerian for $n \geq 3$,
5. bipartite if n is even and

Theorem 1.1: If n is a prime, then the independent domination number of $G(Z_n, \varphi)$ is 1.

Theorem 1.2: The independent domination number of $G(Z_n, \varphi)$ is 2, if $n = 2p$ where p is an odd prime.

Theorem 1.3: Suppose n is neither a prime nor $2p$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots \ldots . . p_k^{\alpha_k}$, where $p_1, p_2, \ldots p_k$ are primes and $\alpha_1, \alpha_2, \ldots . \alpha_k$ are integers ≥ 1, then the independent domination number of $G(Z_n, \varphi)$ is $\frac{n}{p_k}$.

Arithmetic V_n graph

Vasumathi and Vangipuram [4] introduced the concept of Arithmetic V_n graphs and studied some of its properties.

Let n be a positive integer such that $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots \ldots . . p_k^{\alpha_k}$. Then the Arithmetic V_n graph is defined as the graph whose vertex set consists of the divisors of n and two vertices

Theorem 1.4: If $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots \ldots . . p_k^{\alpha_k}$, where $p_1, p_2, \ldots p_k$ are primes and $\alpha_1, \alpha_2, \ldots . \alpha_k$ are integers ≥ 1, then the domination number of $G(V_n)$ is given by

$$\gamma_i(G(V_n)) = \begin{cases} k-1 & \text{if } \alpha_i = 1 \text{ for more than one i} \\ k & \text{Otherwise.} \end{cases}$$

where k is the core of n.

Direct Product Graph $G_1 \times G_2$

In the literature, the direct product is also called as the tensor product, categorical product, cardinal product, relational product, Kronecker product, weak direct product, or conjunction. As an operation on binary relations, the tensor product was introduced by Alfred North Whitehead and Bertrand Russell in their Principia Mathematica[5]. It is also equivalent to the Kronecker product of the adjacency matrices of the graphs given by Weichsel [6].

If a graph can be represented as a direct product, then there may be multiple different representations (direct

2. RESULTS

Let G_1 be an Euler Totient Cayley graph and G_2 be an Arithmetic V_n graph. Then G_1 and G_2 are simple graphs as they have no loops and multiple edges. Hence by the definition of adjacency in direct product, $G_1 \times G_2$ is also a simple graph.

Now we investigate results related to independent domination number of Direct product graphs of Euler totient Cayley graphs and Arithmetic V_n graphs.

Theorem 2.1: If n is a prime, then the independent domination number of $G_1 \times G_2$ is n.

Proof: Let n be a prime. Then $G_1 \times G_2$ is a completely disconnected graph. So there are no edges between these n vertices and its dominating set consists of n isolated

6. complete graph if n is a prime.

The independent domination number of these graphs are studied by the authors [3] and the following results are required and they are presented without proofs.

u, v are adjacent in V_n graph if and only if GCD $(u, v) = p_i$, for some prime divisor p_i of n.

In this graph vertex 1 becomes an isolated vertex. Hence we consider Arithmetic graph V_n without vertex 1 as the contribution of this isolated vertex is nothing when the properties of these graphs and enumeration of some domination parameters are studied.

Clearly, V_n graph is a connected graph. Because when n is a prime, V_n graph consists of a single vertex. Hence it is a connected graph. In other cases, by the definition of adjacency in V_n, there exist edges between prime number vertices and their prime power vertices and also to their prime product vertices. Therefore each vertex of V_n is connected to some vertex in V_n.

The independent domination number of these graphs are obtained by the authors and the proof of the following theorem can be found in [3].

products do not satisfy unique factorization) but each representation has the same number of irreducible factors. Wilfried Imrich [7] gives a polynomial time algorithm for recognizing tensor product graphs and finding a factorization of any such graph.

Let G_1 and G_2 be two simple graphs with their vertex sets as $V_1 = \{u_1, u_2, \ldots, u_l\}$ and $V_2 = \{v_1, v_2, \ldots, v_m\}$ respectively. Then the direct product of these two graphs denoted by $G_1 \times G_2$ is defined to be a graph with vertex set $V_1 \times V_2$, where $V_1 \times V_2$ is the Cartesian product of the sets V_1 and V_2 such that any two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1 \times G_2$ are adjacent if $u_1 u_2$ is an edge of G_1 and $v_1 v_2$ is an edge of G_2.

The cross symbol $\times$, shows visually the two edges resulting from the direct product of two edges.

vertices and these n isolated vertices form a minimum independent dominating set. Hence independent domination number of $G_1 \times G_2$ is n. ■

Theorem 2.2: If $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots \ldots . . p_k^{\alpha_k}$, where $\alpha_i \geq 1$, then the independent domination number of $G_1 \times G_2$ is $\gamma_i(G_1 \times G_2) = \frac{n}{p_k} \cdot |V_2|$.

Proof: Let V_1, V_2 and V denote the vertex sets of the graphs G_1, G_2 and $G_1 \times G_2$ respectively.

By Theorem 1.3 in [3], it is clear that the elements of the set V_1 can be divided into disjoint sets $D_i = \{r.p_k + i \ / \ r = 0, 1, 2, \ldots\ldots, npk-1$ for $1 \leq i \leq pk$, such that each Di is an independent dominating set of G_1 with minimum cardinality $\frac{n}{p_k}$.

CONFERENCE PAPER
National Conference dated 27-28 July 2017 on
Recent Advances in Graph Theory and its Applications (NCRAGTA2017)
Organized by Dept of Applied Mathematics
Sri Padmawati Mahila Vishvavidyalayam (Women's University) Tirupati, A.P.,

In particular, let $D_1 = \{0,\ p_k,\ 2p_k,\ \ldots\ldots,$ $npk-1\ pk$ be an independent dominating set of $G1$ with minimum cardinality $\frac{n}{p_k}$.

Let $V_2 = \{v_1,\ v_2, \ldots\ldots, v_q\}$ be the vertex set of G_2 and consider $D_1 \times V_2$ in $G_1 \times G_2$ as

$$D_i' = D_1 \times V_2$$
$$= \left\{0,\ p_k, 2p_k, \ldots\ldots, \left(\frac{n}{p_k} - 1\right) p_k\right\} \times \{v_1,\ v_2, \ldots\ldots, v_q\}$$

$$= \{\ (0,\ v_1),\quad (0,\ v_2),\quad \ldots\ldots\ldots\ldots\quad (0, v_q),$$
$$(p_k,\ v_1),\quad (p_k,\ v_2),\quad \ldots\ldots\ldots\ldots\quad (p_k, v_q),$$
$$\vdots \qquad \vdots$$
$$\ldots\ldots\ldots\ldots \qquad \vdots$$
$$\left(\left(\frac{n}{p_k} - 1\right) p_k,\ v_1\right),\ \left(\left(\frac{n}{p_k} - 1\right) p_k,\ v_2\right),\ \ldots\ldots,\left(\left(\frac{n}{p_k} - 1pk, vq\right\}\right.\right.$$

Let $x, y \in D_i'$. Then $x = (mp_k, v_i)$ and $y = (lp_k, v_j)$ for $m \neq l,\ i \neq j$. Now x is not adjacent to y because vertex mp_k is not adjacent to vertex np_k as $\mathrm{GCD}(mp_k - lp_k,\ n) = ((m-l)p_k, n) \neq 1$. This shows that no two vertices in D_i' are adjacent. So, D_i' becomes an independent set of $G_1 \times G_2$.

Next we prove that D_i' is a dominating set.

Let (u, v) be any vertex of $V - D_i'$ in $G_1 \times G_2$. Then u is adjacent to at least one sp_k, $0 \leq s \leq \frac{n}{p_k} - 1$, in D_1, as D_1 is a dominating set of G_1. Now suppose v is adjacent to some vertex v_l in V_2. (Certainly there exists such a vertex in V_2 as V_2 has no isolated vertices). Then vertex (u, v) is adjacent to vertex (sp_k, v_l). Hence every vertex (u, v) of $V - D_i'$ in $G_1 \times G_2$ is adjacent to at least one vertex in D_i'. Thus D_i' becomes a dominating set of $G_1 \times G_2$.

We now prove that D_i' is minimum.

Suppose we remove a vertex say (rp_k, v_i) from D_i'. Then the deleted vertex (rp_k, v_i) is not dominated by any other vertex (sp_k, v_j) of D_i', because for $0 \leq s \leq \frac{n}{p_k} - 1$, GCD $(rp_k - sp_k, \mathrm{n}) \neq 1$. This contradicts that D_i' is a dominating set of $G_1 \times G_2$. Hence D_i' becomes an independent dominating set with minimum cardinality. We cannot get any other independent dominating set with less number of vertices than this, because of the properties of prime numbers.

Hence $\gamma_i(G_1 \times G_2) = |D_i'| = \frac{n}{p_k} \cdot |V_2|$. ■

3. ILLUSTRATIONS

For $n = 6$

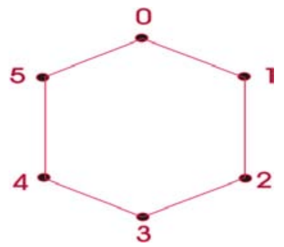

Figure 1
$\boldsymbol{G_1 = G(Z_6, \varphi)}$

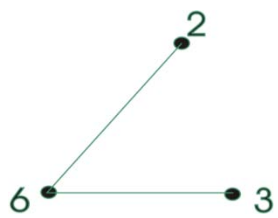

Figure 2
$\boldsymbol{G_2 = G(V_6)}$

CONFERENCE PAPER
National Conference dated 27-28 July 2017 on
Recent Advances in Graph Theory and its Applications (NCRAGTA2017)
Organized by Dept of Applied Mathematics
Sri Padmawati Mahila Vishvavidyalayam (Women's University) Tirupati, A.P.,

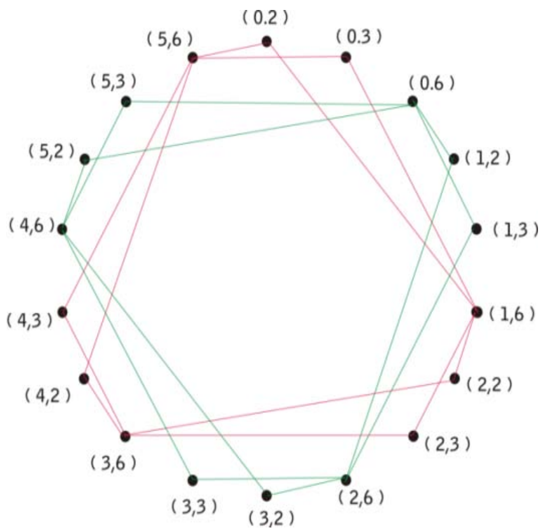

Figure 3

$G_1 \times G_2$

Minimum Independent Dominating set: {(0,2), (0,3), (0,6), (3,2), (3,3), (3,6)}

For n = 8

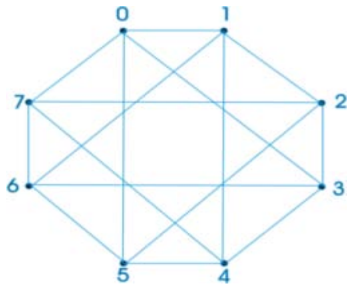

Figure 4

$G_1 = G(Z_8, \varphi)$

Figure 5

$G_2 = G(V_8)$

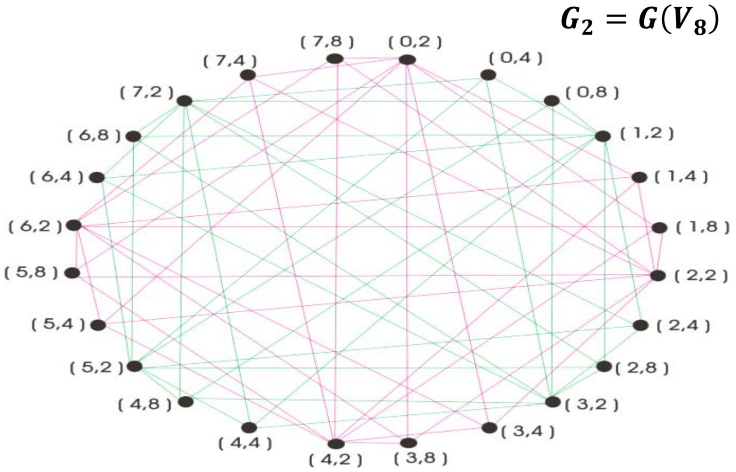

Figure 6

$G_1 \times G_2$

Minimum Independent Dominating set: {(0,2), (0,4), (0,8), (2,2), (2,4), (2,8), (4,2), (4,4), (4,8), (6,2), (6,4), (6,8)}

For n = 11

CONFERENCE PAPER
National Conference dated 27-28 July 2017 on
Recent Advances in Graph Theory and its Applications (NCRAGTA2017)
Organized by Dept of Applied Mathematics
Sri Padmawati Mahila Vishvavidyalayam (Women's University) Tirupati, A.P.,

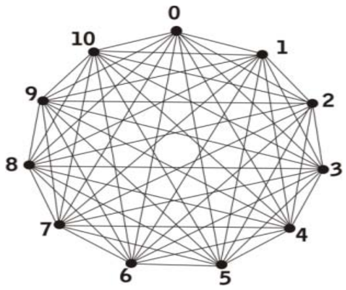

Figure 7
$\boldsymbol{G_1 = G(Z_{11}, \varphi)}$

Figure 8
$\boldsymbol{G_2 = G(V_{11})}$

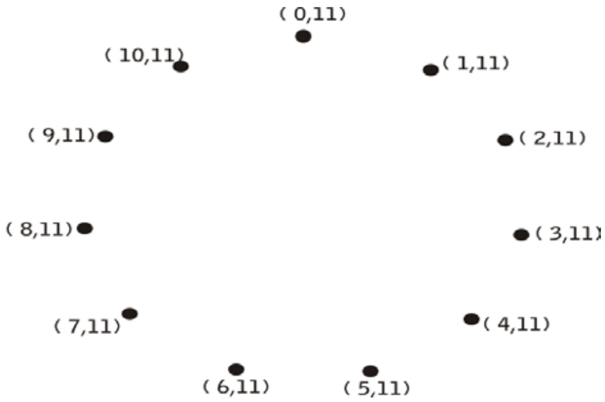

Figure 9
$\boldsymbol{G_1 \times G_2}$
Minimum Independent Dominating set: Vertex set of $G_1 \times G_2$

CONCLUSION

Graph Theory is young but rapidly maturing subject. Its basic concepts are simple and can be used to express problems from many different subjects. The purpose of this work is to familiarize the reader with the direct product graph of Euler Totient Cayley graph with Arithmetic Vn graph. It is useful other Researchers for further studies of other properties of these product graphs and their relevance in both combinatorial problems and classical algebraic problems.

5. REFERENCES

[1] Nathanson and Melvyn B . Connected components of arithmetic graphs, Monat. fur. Math, 29, 219 – 220(1980).
[2] Madhavi, L.-Studies on domination parameters and enumeration of cycles in some Arithmetic Graphs, Ph.D. Thesis submitted to S.V.University, Tirupati, India, (2002).
[3] Uma Maheswari S and Maheswari B: Independent domination number of Euler totient Cayley graphs and arithmetic graphs, IJARET**,** Volume 7, Issue 3, 56–65, May–June (2016)
[4] Vasumathi, N. - Number theoretic graphs, Ph. D. Thesis submitted to S.V.University, Tirupati, India, (1994).
[5] Whitehead, A.N. and Russel, B. Principia Mathematica, Volume 2, Cambridge University, Press, Cambridge (1912).
[6] Weichsel, P.M. The Kronecker product of graphs, Proc. Amer. Math.Soc., 13, 47-52, (1962).
[7] Imrich, W. and Klavzar, S. - Product graphs: Structure and recognition, John, Wiley & Sons, New York, USA (2000).

CONFERENCE PAPER
National Conference dated 27-28 July 2017 on
Recent Advances in Graph Theory and its Applications (NCRAGTA2017)
Organized by Dept of Applied Mathematics
Sri Padmawati Mahila Vishvavidyalayam (Women's University) Tirupati, A.P.,