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Fixed Point Theorems in 2-Normed Spaces by Sub-additive Altering Distance Function

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Abstract: the study of fixed point theorem by applying the concept of 2-normed space has been initiated in [1]" M. Kır, K. Hukmi, Some New Fixed Point Theorems in 2-Normed Spaces,Int. Journal of Math. Analysis, vol. 7, pp. 2885 – 2890, 2013". Using the idea of altering distance function with the property of subadditivity and the 2-normed spaces, we have defined a fixed point and established some results assuring the existence and uniqueness of fixed point with reference to different conditions.

Keywords: 2-Normed Space, 2- Banach Spaces, Fixed Point, Sub-additive altering distance Function.

AMS CLASSIFICATION: 47H10; 41A65; 41A15.

1. INTRODUCTION

Fixed point theory is one of the most important component in the development of the functional analysis. Also, it has been used effectively in many branches of science and technology, such as chemistry, biology, economics, computer science, engineering and so on. Recently Harikrishnan and Ravindran [2] introduced some new properties of accretive mappings and contraction mappings in 2-normed spaces. Mehmet kir and Hukmi[1] introduced some basic definitions and established results in 2-normed spaces. Kannan [3] has initiated Kannan mappings for new contractive condition. A similar contractive condition has been introduced by Chatterjea [4] for complete metric space and introduced a new type mapping called Chatterjea mapping.

By using the concept of altering distance function, many mathematician solved the fixed point problems during last decade by Choudhury [5] and Dutta [6]. Pawar and Sahu [7] established new result for the existence of unique fixed point theorem using sub-additive altering distance function.

In the present paper, we establish a fixed point theorem for 2-normed spaces using sub-additive altering distance function for asserting the existence and uniqueness of fixed point.

2. PRELIMINARIES

In this section, we recall some definitions and concepts which are required for our analysis.

Definition 2.1[7]A function $\phi: [0, \infty) \to [0, \infty)$is called a **sub-additivealtering distance function** if the following properties are satisfied:

(i) $\phi$is a continuous function

(ii) ϕ is a monotonically increasing function

(iii) ϕ(x) = 0 $\Leftrightarrow$ x = 0

(iv) $\phi(x + y) \leq \phi(x) + \phi(y),\ \forall\, x, y \in [0, \infty)$.

Example 2.1 A function $\phi: [0, \infty) \to [0, \infty)$is defined as $\phi(x) = 3x$.

Definition 2.2 [8]Let E be a real linear space with $dim\, E \geq 2$ and $\|\ ,\|: E\ X\ E \to [0, \infty)$ be a function. Then, $(E, \|\ ,\|)$ is called a linear 2- normed space for all $x, y, z \in E$ and $\alpha \in R$,

i.) $\|\ x, y\ \| = 0$if and only if x and y are linearly dependent.

ii.) $\|\ x, y\ \| = \|\ y,\ x\ \|$

iii.) $\|\alpha x\ , y\| = |\alpha|\|x,\ y\|$

iv.) $\|\ x, y\ \| \leq \|\ x,\ z\ \| + \|\ z, y\ \|$

Throughout our discussions, we denote $E = R^2$.

Example 2.2$\|\ ,\ \|: E\ X\ E \to [0, \infty)$ is defined as

$\|\ (x_1, y_1)\ , (x_2, y_2)\ \| = |\ x_1 y_2 - y_1 x_2|$is 2-normed space.

Definition 2.3 [8] *A* sequence$\{x_n\}$in a 2-normed space$(E, \|.\,,.\|)$ is said tobe a Cauchy sequence,

If $\lim_{m,n\to\infty} \|x_n - x_m, z\| = 0$for all$z \in E$.

Example 2.3 Consider a 2-normed space defined in the Example 2.2. It may be verified easily that the sequence$\{x_n\} = \{(\frac{1}{n}, 0)\}$is a Cauchy sequence in this space.

Definition 2.4 [8] *A* sequence$\{x_n\}$in a 2-normed space$(E, \|.\,,.\|)$ is said to be convergentif there is a point x in E such thatlim $n \to \infty\ \|x_n - x, z\| = 0$ for all$z \in E$. *If* $\{x_n\}$ converges to x we write$x_n \to x\ as\ n{\to}\infty$.

Example 2.4 Again consider a 2-normed space defined in the Example 2.2. It may be verified easily that the sequence $\{x_n\} = \{(\frac{1}{n+1}, 0)\}$is a Cauchy sequence in this space.

Definition 2.5 [8]A linear 2-normed space is said to be complete if everyCauchy sequence is convergent to an element of E. A complete 2-normedspace E is called 2-Banach space.

Example 2.5 R^2 is a complete 2-normed space with $\|\ (x_1, y_1)\ , (x_2, y_2)\ \| = |\ x_1 y_2 - y_1 x_2|$.

Definition 2.6[2]Let (E, ‖.,.‖) be a linear 2-normed space. Then the mapping T : E $\to$ E is said to be a contraction if there exists k $\in$ [0, 1) such that

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$\|Tx - Ty, z\| \le k\, \|x - y, z\|, \quad \forall\, x, y, z \in E.$

Example 2.6 Let $E = R^2$ and $\|\ ,\| : E\ X\ E \to [0, \infty)$ is defined as

$\|\ (x_1, y_1), (x_2, y_2)\ \| = |\, x_1 y_2 - y_1 x_2 |$. A mapping $T:\ E \to E$ defined as $T(x, y) = (\frac{x}{2}, \frac{y}{2})$ is a contraction mapping.

3. MAIN RESULTS

Applying the concept of altering distance function to contraction maps [11], in this section, we have established the existence of fixed point in 2-Banach Space. We now discuss our two main results.

Theorem 3.1: Let $(X, \|.,.\|)$ be a linear 2-Banach space and K be a nonempty closed and bounded subset of X. If $T: K \to K$ satisfying

$\phi[\|Tx - Ty, z\|] \le \alpha\phi[\|x - Tx, z\| + \|y - Ty, z\|] \qquad (3.1)$

where $\alpha \in [0, \frac{1}{2})$ and ϕ is a sub-additive altering distance function, then T has a unique fixed point in K.

Proof: Let x_0 be an arbitrary point in K and $\{x_n\}$ be a sequence of points of K.

Consider $x_n = Tx_{n-1}, \ \forall\, n = 1,2,3 \ldots.$

Now, $\qquad \phi\|x_n - x_{n+1}, z\| = \phi\|Tx_{n-1} - Tx_n, z\|$

$\le \alpha\phi[\|x_{n-1} - Tx_{n-1}, z\| + \|x_n - Tx_n, z\|]$ (by the eq. (3.1))

$\le \alpha\phi[\|x_{n-1} - x_n, z\| + \|x_n - x_{n+1}, z\|]$

$\le \alpha\phi[\|x_{n-1} - x_n, z\|] + \alpha\phi[\|x_n - x_{n+1}, z\|]$(by the sub-additive of the ϕ .)

$\phi\|x_n - x_{n+1}, z\| - \alpha\phi[\|x_n - x_{n+1}, z\| \le \alpha\phi[\|x_{n-1} - x_n, z\|]$

$\phi\|x_n - x_{n+1}, z\| \le \frac{\alpha}{1-\alpha}\phi[\|x_{n-1} - x_n, z\|]$

We next consider $\phi\|x_{n-1} - x_n, z\|$ and by repeating the same procedure, we may show that

$$\phi\|x_{n-1} - x_n, z\| \le \frac{\alpha}{1-\alpha}\phi[\|x_{n-2} - x_{n-1}, z\|]$$

We continue the process n- times and arrive at

$\phi\|x_n - x_{n+1}, z\| \le \gamma^n\phi[\|x_0 - x_1, z\| \qquad (3.2)$

Where, $\gamma = \frac{\alpha}{1-\alpha} \in [0,1)$.

Claim: $\{x_n\}$ is a Cauchy sequence.

Using the sub-additive property of , if may be seen easily that

$\phi\|x_n - x_m, z\| \le \phi\|x_n - x_{n+1}, z\| + \phi\|x_{n+1} - x_{n+2}, z\| + \cdots + \phi\|x_{m-1} - x_m, z\|$.

By the eq. (3.2)

$$\le \gamma^n\phi[\|x_0 - x_1, z\|] + \gamma^{n+1}\phi[\|x_0 - x_1, z\|] + \cdots + \gamma^{m-1}\phi[\|x_0 - x_1, z\|]$$

$\le [1 + \gamma + \gamma^2 + \cdots + \gamma^{m-1-n}]\,\gamma^n\phi[\|x_0 - x_1, z\|]$.

Letting $m, n \to \infty$, we have $\lim_{m,\ n\to\infty} \phi\|x_n - x_m, z\| = 0$. Thus $\{x_n\}$is a Cauchy sequence in K.

Since K is Complete so $\{x_n\}$is convergent. Hence $\exists\ x^* \in K$ such that $x_n \to x^*\ \ as\ n \to \infty$.

Now, we shall show that $x^* \in K$ is a Fixed Point of T.

$$\phi\|x^* - Tx^*, z\| \le \phi\|x^* - x_n, z\| + \phi\|x_n - Tx^*, z\|$$

$$\le \phi\|x^* - x_n, z\| + \phi\|x_n - x_{n+1}, z\| + \phi\|x_{n+1} - Tx^*, z\|$$

$$\le \phi\|x^* - x_n, z\| + \phi\|x_n - x_{n+1}, z\| + \phi\|\mathrm{T}x_n - Tx^*, z\|$$

By the eq (3.1) & (3.2)

$$\le \phi\|x^* - x_n, z\| + \gamma^n\,\phi\|x_0 - x_1, z\| + \alpha\phi\|x_n - Tx_n, z\| + \alpha\phi\|x^* - Tx^*, z\|$$

Letting $n \to \infty$,

$\phi\|x^* - Tx^*, z\| \le 2\alpha\phi\|x^* - Tx^*, z\| \qquad (3.3)$

Since $\alpha < \frac{1}{2}$, (3.3) leads to a contradiction. Hence,

$\phi\|x^* - Tx^*, z\| = 0 \Rightarrow \|x^* - Tx^*, z\| = 0$ (by def. 2.1)

$$\Rightarrow x^* = Tx^*$$

Claim: $x^*\ is$ unique fixed point .

Assume that, if possible $x' \in K$ is another fixed point of T. Hence, we have $Tx' = x'$.

Now, $\qquad \phi\|x^* - x', z\| = \phi\|Tx^* - Tx', z\|$

$$\le \alpha\phi[\|x^* - Tx^*, z\| + \|x' - \mathrm{T}x', z\|]$$

$$\le \alpha\phi[\|0, z\| + \|0, z\|$$

$$\le \alpha\phi[0]$$

$$\le 0$$

Since ,ϕ is a non-negative function, therefore

$$\phi\|x^* - x', z\| = 0 \Rightarrow \|x^* - x', z\| = 0$$

$\Rightarrow \quad x^* - x' = 0 \quad \forall\, z \in K$

$\Rightarrow \quad x^* = x'$.

This completes the proof.

Theorem 3.2: Let $(X, \|.,.\|)$be a linear 2-Banach space and K be a nonempty closed and bounded subset of X. If $T: K \to K$ satisfying

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$\phi[\|Tx - Ty, z\|] \leq \beta\phi[\|x - Ty, z\| + \|y - Tx, z\|]$ (3.4)

where $\beta \in [0, \frac{1}{2})$ and ϕ is a sub-additive altering distance function, then T has a unique fixed point in K.

Proof: Let x_0 be an arbitrary point in K and $\{x_n\}$ be a sequence of points of K.

Consider $x_n = Tx_{n-1}, \forall\, n = 1,2,3 ....$

Now, $\phi\|x_n - x_{n+1}, z\| = \phi\|Tx_{n-1} - Tx_n, z\|$ (by eq. (3.4))

$$\leq \beta\phi[\|x_{n-1} - Tx_n, z\| + \|x_n - Tx_{n-1}, z\|]$$

$$\leq \beta\phi[\|x_{n-1} - x_{n+1}, z\|$$

$$\leq \beta\phi[\|x_{n-1} - x_n, z\| + \|x_n - x_{n+1}, z\|]$$

$$\leq \beta\phi[\|x_{n-1} - x_n, z\|] + \beta\phi[\|x_n - x_{n+1}, z\|]$$

(by the sub additive of the ϕ .)

$\phi\|x_n - x_{n+1}, z\| - \beta\phi[\|x_n - x_{n+1}, z\| \leq \beta\phi[\|x_{n-1} - x_n, z\|]$

$$\phi\|x_n - x_{n+1}, z\| \leq \frac{\beta}{1-\beta}\phi[\|x_{n-1} - x_n, z\|]$$

We next consider $\phi\|x_{n-1} - x_n, z\|$ and by repeating the same procedure, we may show that

$$\phi\|x_{n-1} - x_n, z\| \leq \frac{\beta}{1-\beta}\phi[\|x_{n-2} - x_{n-1}, z\|]$$

We continue the process n- times and arrive at

$\phi\|x_n - x_{n+1}, z\| \leq \delta^n\phi[\|x_0 - x_1, z\|]$ (3.5)

Where, $\delta = \frac{\beta}{1-\beta} \in [0,1)$.

Claim: $\{x_n\}$ is a Cauchy sequence.

Using the subadditive property of, if may be seen easily that

$\phi\|x_n - x_m, z\| \leq \phi\|x_n - x_{n+1}, z\| + \phi\|x_{n+1} - x_{n+2}, z\| + \cdots + \phi\|x_{m-1} - x_m, z\|$.

By eq. (3.5),

$$\leq \delta^n\phi[\|x_0 - x_1, z\|] + \delta^{n+1}\phi[\|x_0 - x_1, z\|] + \cdots + \delta^{m-1}\phi[\|x_0 - x_1, z\|]$$

$\leq [1 + \delta + \delta^2 + \cdots + \delta^{m-1-n}]\, \delta^{nr}\phi[\|x_0 - x_1, z\|]$.

Letting $m, n \to \infty$, we have $\lim_{m,\ n\to\infty}\phi\|x_n - x_m, z\| = 0$. Thus $\{x_n\}$ is a Cauchy sequence in K. Since K is Complete so $\{x_n\}$ is convergent sequence. Hence $\exists\, x^* \in K$ such that $x_n \to x^*$ $as\, n \to \infty$.

Now, we will show that $x^* \in K$ is a Fixed Point of T.

$$\phi\|x^* - Tx^*, z\| \leq \phi\|x^* - x_n, z\| + \phi\|x_n - Tx^*, z\|$$

$$\leq \phi\|x^* - x_n, z\| + \phi\|x_n - x_{n+1}, z\| + \phi\|x_{n+1} - Tx^*, z\|$$

By eq. (3.3) & (3.4)

$$\leq \phi\|x^* - x_n, z\| + \delta^n\, \phi\|x_0 - x_1, z\| + \beta\phi\|x_n - Tx^*, z\| + \beta\phi\|x^* - Tx_n, z\|$$

Letting $n \to \infty$,

$\phi\|x^* - Tx^*, z\| \leq 2\beta\phi\|x^* - Tx^*, z\|$ (3.6)

Since $\beta < \frac{1}{2}$, (3.6) leads to a contradiction. Hence,

$$\phi\|x^* - Tx^*, z\| = 0 \Rightarrow \|x^* - Tx^*, z\| = 0$$

$$\Rightarrow x^* = Tx^*$$

Claim: $x^*\ is$ unique fixed point .

Assume that, if possible $x' \in K$ is another fixed point of T. Hence, we have $Tx' = x'$.

Now , $\phi\|x^* - x', z\| = \phi\|Tx^* - Tx', z\|$

$$\leq \beta\phi[\|x^* - Tx', z\| + \|x' - Tx^*, z\|]$$

$\leq \beta\phi[\|0, z\| + \|0, z\|]$

$$\leq \beta\phi[0]$$

$$\leq 0$$

Since ,ϕ is a non-negative function, therefore

$$\phi\|x^* - x', z\| = 0 \Rightarrow \|x^* - x', z\| = 0$$

$\Rightarrow \quad x^* - x' = 0 \quad \forall\, z \in K$

$\Rightarrow \quad x^* = x'$.

This completes the proof.

APPLICATION

The concept of fixed point in 2- Normed space may be useful in Computer science, Physics, Biology etc.It can be applied, where two or more equations work simultaneously and solution is required. An example if any place two equations whose solutions are irrational .The approximate solution of equations will be a fixed point. By finding fixed point of these equations we can find the required solution.

Example: Suppose a machine have pair of condition $(x^3 - 3, x^2 - 2)$. The approximate solution of equations is (1.44224957, 1.41421356) .

REFERENCES

[1] M. Kır, K. Hukmi, “Some new fixed point Theorems in 2-Normed Spaces,” Int. Journal of Math. Analysis, vol. 7, pp. 2885 – 2890, 2013.

[2] P. K Harikrishnan, K. T. Ravindran, “Some properties of accretive operators in linear 2-Normed spaces,” International Mathematical Forum, vol. 6, pp. 2941-2947, 2011.
[3] R. Kannan, “Some results on fixed points,” Bull. Calcutta Math. Soc., vol. 60, pp. 71-76, 1968.
[4] S. K. Chatterjea, “Fixed point theorems,” C. R. Acad. Bulgare Sci. vol. 25, pp. 727-730, 1972.
[5] B. S. Choudhury, “A common unique fixed point result in metric spaces involving generalized altering distances,” Math. Commun., vol. 10, pp. 105-110, 2005.
[6] B. S. Choudhury, and P. N. Dutta, “A unified fixed point result in metric spaces involving a two variable function,” Filomat, vol. 14, pp. 43-48, 2000.
[7] P.L. Powar, G. R. K. Sahu, “Common Fixed Point Theorem by Subadditive Altering Distance Function for Sequence of Mappings,” Scientific & Academic Publishing, vol. 3 pp. 190-193, 2013.
[8] R. W. Freese, Y. J. Cho, Geometry of linear 2-normed space, Huntington N. Y. Nova Puplishers, 2001.

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