



Multi Criteria Decision Making Based on Classical α -Cut Approach

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Abstract: Multi Criteria Decision Making (MCDM) is the process to find the best alternatives among a set of available alternatives in presence of various conflicting criteria. In this paper MCDM method is presented using α -cut approach and it is observed that α -cut approach is general enough to deal with MCDM methods. Here first fuzzy numbers have been assigned to quantify different qualitative criteria of alternatives and then their α -cut values are evaluated and then performed some arithmetic operation to order ranking of the alternatives. Finally, a decision making problem is carried out with different case studies by using different shapes of fuzzy numbers.

Keywords: Multi criteria decision making, fuzzy numbers, α -cut of fuzzy number.

1. INTRODUCTION

Crisp data are not always adequate to model in many real life situations where as fuzzy set theory is more suitable to handle such type of situation. Very often in MCDM problems, data are imprecise and so fuzzy set come into picture. For representation of fuzziness in the decision data and group decision making process and to quantify the qualitative factors, linguistic variables are used to access the weight of all criteria and ratings of each alternative with respect to each criterion. These linguistic terms are further translated into mathematical measure by using different fuzzy numbers. The nature of selection process is a complex multi-attribute group decision making problem which deals with both quantitative and qualitative factors may be conflicting in nature as well as contain incomplete and uncertain information. Generally, triangular and trapezoidal fuzzy membership functions are widely studied in literature to represent the uncertainty. In our present study we have used the bell shaped fuzzy number and triangular fuzzy numbers to specify the qualitative factors. Different arithmetic operation of bell shape fuzzy numbers and triangular fuzzy numbers are performed and have applied by associating with α -cut approach in MCDM problems.

The fuzzy set theory was introduced by L.A. Zadeh in 1965 [2] to deal with problems in which a source of vagueness is involved. Bellman and Zadeh in 1970 [3] and Zimmermann in 1987 [4] took the fuzzy sets to deal with problems associated with vague, imprecise and ill concept into standard multi criteria decision making (MCDM) techniques and also proposed that the constraints related to a MCDM problem could be defined as fuzzy sets in the space of alternatives. Many MCDM methods are available in literature, some of them are the fuzzy analytic hierarchy process (AHP), fuzzy technique for order preference by similarity to ideal solution (TOPSIS), elimination and choice translating reality (ELECTRE), VIKOR Method, it stands for 'Vlsekriterijumska Optimizacija I Kompromisno Resenje', means multi-criteria optimization and compromise solution was developed by Opricovic and preference ranking organization method for enrichment evaluation

(PROMETHEE). Baas and Kwakernaak in 1977 [5] works on classical MCDM methods. Kickert (1978) [6], Zimmermann's (1985) [25], Chen and Hwang(1992) [8]; Fodor and Roubens(1994) [9]; Luhandjula (1989) [12]; Sakawa (1993) [15], Ribeiro (1996) [14], Ravi and Reddy (1999) [13], Fan et al. (2002, 2004) [16,17], Wang and Parkan (2005)[18], Omero et al. (2005) [20], Hua et al. (2005) [21], Ling (2006) [22], Xu and Chen (2007) [19], have developed many approaches for fuzzy multi criteria decision making methods. Lin & Chen (2004) [23] developed a fuzzy linguistic approach for bid decision making process. Li.et. (2005) [24] propose a multi layer fuzzy pattern recognition method for selection of contractor. D.Singh et.al. (2005) [1] propose a fuzzy decision frame work for alternatives selection. The increasing numbers of studies have dealt with uncertain fuzzy problems by applying the fuzzy set theory extensively to help solving the MCDM problems (Liou & Chen, 2006 [28]; Benitez et al., 2007 [29]; Shipley & Coy, 2009 [33]; Parameshwaran et al., 2009 [30]; Rahman & Qureshi, 2009 [31]; Büyüközkan, 2010 [32]). Many researchers have applied MCDM to find the best remanufacturing technology (Wadhwa, Madaan et al. 2009 [35], Jiang, Zhang et al. 2011 [36]), while others have investigated the importance of factors (Subramoniam, Huisingh et al. 2013 [37], Tian, Chu et al. 2014 [39]) or barriers (Zhu, Sarkis et al. 2014 [38]) affecting remanufacturing processes, in natural resource management (Mendoza and Martins 2006 [40]), and in construction (Jato-Espino, Castillo-Lopez et al. 2014 [41]) in supplier evaluation and selection (Ho, Xu et al. 2010 [42])

In this study, D. Singh and Robert L.K. Tiong's MCDM method has been considered and is interpreted this method with α -cut approach by performing different arithmetic operation of fuzzy numbers and their fusions. This method allows us to use the any fuzzy number to represent the uncertainty because α cut of any fuzzy number can be obtained very easily with the help of membership functions by taking α values between 0 and 1.

2. FUZZY SET THEORY

In this section some necessary backgrounds and notation of fuzzy set theory are reviewed.

Definition 2.1:[26] Let X be a space of points. A fuzzy set A in X is characterized by a membership function $\mu_A(x)$ which associates with each point in X a real number in the interval $[0, 1]$ i.e., $\mu_A(x) : X \rightarrow [0, 1]$, with the value of $\mu_A(x)$ at x representing the grade of membership of x in A .

Definition 2.2:[34] The height of a fuzzy set is the largest membership grade attained by any element in that set. A fuzzy set A in the universe of discourse X is called the normalized when the height of A is equal to one.

Definition 2.3:[34] A fuzzy set A in a universe of discourse X is convex if and only if $\mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$ for all x_1, x_2 in X and all $\lambda \in [0, 1]$ where $\min$ denotes the minimum operator.

Definition 2.4:[11] Given a fuzzy set A in X and any real number $\alpha \in [0, 1]$ then the α -cut of A , denoted by ${}^\alpha A$ is the crisp set ${}^\alpha A = \{x \in X : \mu_A(x) \geq \alpha\}$. The strong α -cut, denoted by ${}^{\alpha+} A$ is the crisp set ${}^{\alpha+} A = \{x \in X : \mu_A(x) > \alpha\}$.

Definition 2.5:[11] A fuzzy number is a convex normalized fuzzy set of the real line R whose membership function is piecewise continuous.

Definition 2.6:[11] A triangular fuzzy number A can be defined as a triplet $[a, b, c]$. Its membership function is defined as:

$$\mu_A(x) = \begin{cases} \frac{x-a}{b-a} & ; a \leq x \leq b \\ \frac{c-x}{c-b} & ; b \leq x \leq c \end{cases}$$

Definition 2.7:[26] Gaussian Fuzzy Number : The membership function of a Gaussian fuzzy number is defined as

$$\mu_A(x) = \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right), \text{ where } \mu \text{ and } \sigma \text{ represent the MFs centre and MFs width.}$$

Definition 2.8:[26] Cauchy Fuzzy Number : The membership function of Cauchy fuzzy number is defined as

$$\mu_A(x) = \frac{1}{1 + \left(\frac{x-p}{q}\right)^2}; p \text{ and } q \text{ represent the MFs centre and MFs width respectively.}$$

Definition 2.9:[10] The α Cut of a Triangular Fuzzy Number :

Equating $\alpha \in [0, 1]$ to both left and right of the above defined membership function for triangular fuzzy number A we have

$$\alpha = \frac{x-a}{b-a} \text{ and } \alpha = \frac{c-x}{c-b}.$$

Now expressing x in terms of α it gives $\alpha = [(b-a)\alpha + a]$ and $x = c - (c-b)\alpha$ and hence the α cut of A is

$${}^\alpha A = [(b-a)\alpha + a, c - (c-b)\alpha].$$

Definition 2.10:[10] The α Cut of Gaussian Fuzzy Number :

For the membership function $\mu_A(x) = \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$; let $\mu_A(x) \geq \alpha$; $\alpha \in (0, 1)$, we have

$$\exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) \geq \alpha \Rightarrow -(x-\mu)^2 \geq 2\sigma^2 \ln \alpha \Rightarrow -(\sigma\sqrt{-2\ln \alpha}) \leq (x-\mu) \leq \sigma\sqrt{-2\ln \alpha}$$

$$\text{Therefore } \mu - (\sigma\sqrt{-2\ln \alpha}) \leq x \leq \mu + \sigma\sqrt{-2\ln \alpha}$$

$$\text{Thus } {}^\alpha A = [\mu - \sigma\sqrt{-2\ln \alpha}, \mu + \sigma\sqrt{-2\ln \alpha}]$$

Definition 2.11:[26] The α Cut of Cauchy Fuzzy Number :

$$\text{The membership function for Cauchy fuzzy number is } \mu_A(x) = \frac{1}{1 + \left(\frac{x-p}{q}\right)^2}$$

Let $\mu_A(x) = \frac{1}{\left(1 + \left(\frac{x-p}{q}\right)^2\right)} \geq \alpha; \alpha \in (0,1)$, we have

$$\Rightarrow (x-p)^2 \leq q^2 \left(\frac{1-\alpha}{\alpha}\right) \Rightarrow (x-p)^2 \leq \left(q\sqrt{\frac{1-\alpha}{\alpha}}\right)^2 \Rightarrow p - q\sqrt{\frac{1-\alpha}{\alpha}} \leq (x-p) \leq p + q\sqrt{\frac{1-\alpha}{\alpha}}$$

$$\text{Thus } {}^\alpha A = \left[p - q\sqrt{\frac{1-\alpha}{\alpha}}, p + q\sqrt{\frac{1-\alpha}{\alpha}} \right].$$

Definition 2.12 : Defuzzified value: For a Gaussian fuzzy number

$\tilde{m} = [\mu - \sigma\sqrt{-2\ln\alpha}, \mu + \sigma\sqrt{-2\ln\alpha}]$, defuzzified value is defined as the mean of the Gaussian fuzzy numbers. That is

$$\text{defuzzified value} = \frac{\mu - \sigma\sqrt{-2\ln\alpha} + \mu + \sigma\sqrt{-2\ln\alpha}}{2} = \mu.$$

3. ARITHMETIC OPERATION OF DIFFERENT FUZZY NUMBERS USING A-CUT METHOD

A. Gaussian and Triangular fuzzy numbers:

The general form of the α cut of Gaussian fuzzy number is

$${}^\alpha A_1 = [\mu - \sigma\sqrt{-2\ln\alpha}, \mu + \sigma\sqrt{-2\ln\alpha}]; \mu \text{ and } \sigma \text{ represent the MFs centre and MFs width respectively and } \alpha \in (0,1).$$

The general form of the α cut of triangular fuzzy number (a,b,c) is given by

$${}^\alpha A_2 = [(b-a)\alpha + a, c - (c-b)\alpha]$$

3. A.1. Addition:

$${}^\alpha A_1 + {}^\alpha A_2 = [\mu - \sigma\sqrt{-2\ln\alpha} + (b-a)\alpha + a, \mu + \sigma\sqrt{-2\ln\alpha} + c - (c-b)\alpha]$$

3. A.2. Subtraction:

$${}^\alpha A_1 - {}^\alpha A_2 = [\mu - \sigma\sqrt{-2\ln\alpha} - \{c - (c-b)\alpha\}, \mu + \sigma\sqrt{-2\ln\alpha} - \{(b-a)\alpha + a\}]$$

3. A.3. Multiplication:

$${}^\alpha A_1 \times {}^\alpha A_2 = [\{\mu - \sigma\sqrt{-2\ln\alpha}\} \times \{(b-a)\alpha + a\}, \{\mu + \sigma\sqrt{-2\ln\alpha}\} \times \{c - (c-b)\alpha\}]$$

3. A.4. Division:

$$\frac{{}^\alpha A_1}{{}^\alpha A_2} = \left[\frac{(\mu - \sigma\sqrt{-2\ln\alpha})}{c - (c-b)\alpha}, \frac{(\mu + \sigma\sqrt{-2\ln\alpha})}{((b-a)\alpha + a)} \right]$$

B. Cauchy and Triangular fuzzy numbers:

The general form of α cut of Cauchy fuzzy number is ${}^\alpha A_1 = \left[p - q\sqrt{\frac{1-\alpha}{\alpha}}, p + q\sqrt{\frac{1-\alpha}{\alpha}} \right]; p \text{ and } q \text{ are the MFs centre and MFs width of the Cauchy fuzzy number.}$

The general form of α cut of triangular fuzzy number (a,b,c) is ${}^\alpha A_2 = [(b-a)\alpha + a, c - (c-b)\alpha]$

3. B.1 Addition.

$${}^\alpha A_1 + {}^\alpha A_2 = \left[p - q\sqrt{\frac{1-\alpha}{\alpha}} + (b-a)\alpha + a, p + q\sqrt{\frac{1-\alpha}{\alpha}} + c - (c-b)\alpha \right]$$

3. B.2 Subtraction.

$${}^\alpha A_1 - {}^\alpha A_2 = \left[p - q\sqrt{\frac{1-\alpha}{\alpha}} - \{c - (c-b)\alpha\}, p + q\sqrt{\frac{1-\alpha}{\alpha}} - \{(b-a)\alpha + a\} \right]$$

3. B.3 Multiplication.

$${}^\alpha A_1 \times {}^\alpha A_2 = \left[\left(p - q\sqrt{\frac{1-\alpha}{\alpha}} \right) \times \{(b-a)\alpha + a\}, \left(p + q\sqrt{\frac{1-\alpha}{\alpha}} \right) \times \{c - (c-b)\alpha\} \right]$$

3. B.4 Division.

$$\frac{{}^{\alpha}A_1}{{}^{\alpha}A_2} = \left[\frac{\left(p - q\sqrt{\frac{1-\alpha}{\alpha}} \right)}{\{c - (c-b)\alpha\}}, \frac{\left(p + q\sqrt{\frac{1-\alpha}{\alpha}} \right)}{\{(b-a)\alpha + a\}} \right]$$

C. Cauchy and Gaussian fuzzy numbers.

The general form of α cut of Cauchy fuzzy number is ${}^{\alpha}A_1 = \left[p - q\sqrt{\frac{1-\alpha}{\alpha}}, p + q\sqrt{\frac{1-\alpha}{\alpha}} \right]$; p and q are the MFs centre and

MFs width of the Cauchy fuzzy number.

The general form of the α cut of Gaussian fuzzy number is

${}^{\alpha}A_2 = [\mu - \sigma\sqrt{-2\ln \alpha}, \mu + \sigma\sqrt{-2\ln \alpha}]$; μ and σ represent the MFs centre and MFs width respectively and $\alpha \in (0,1)$

3. C.1 Addition.

$${}^{\alpha}A_1 + {}^{\alpha}A_2 = \left[p - q\sqrt{\frac{1-\alpha}{\alpha}} + \mu - \sigma\sqrt{-2\ln \alpha}, p + q\sqrt{\frac{1-\alpha}{\alpha}} + \mu + \sigma\sqrt{-2\ln \alpha} \right]$$

3. C.2 Subtraction.

$${}^{\alpha}A_1 - {}^{\alpha}A_2 = \left[p - q\sqrt{\frac{1-\alpha}{\alpha}} - \{\mu + \sigma\sqrt{-2\ln \alpha}\}, p + q\sqrt{\frac{1-\alpha}{\alpha}} - \{\mu - \sigma\sqrt{-2\ln \alpha}\} \right]$$

3. C.3 Multiplication.

$${}^{\alpha}A_1 \times {}^{\alpha}A_2 = \left[\left(p - q\sqrt{\frac{1-\alpha}{\alpha}} \right) \times \{\mu - \sigma\sqrt{-2\ln \alpha}\}, \left(p + q\sqrt{\frac{1-\alpha}{\alpha}} \right) \times \{\mu + \sigma\sqrt{-2\ln \alpha}\} \right]$$

3. C.4 Division.

$$\frac{{}^{\alpha}A_1}{{}^{\alpha}A_2} = \left[\frac{\left(p - q\sqrt{\frac{1-\alpha}{\alpha}} \right)}{\mu + \sigma\sqrt{-2\ln \alpha}}, \frac{\left(p + q\sqrt{\frac{1-\alpha}{\alpha}} \right)}{\mu - \sigma\sqrt{-2\ln \alpha}} \right]$$

D. Gaussian and Gaussian fuzzy numbers:

Consider two Gaussian fuzzy numbers ${}^{\alpha}A_1 = [\mu_1 - \sigma_1\sqrt{-2\ln \alpha}, \mu_1 + \sigma_1\sqrt{-2\ln \alpha}]$ and

${}^{\alpha}A_2 = [\mu_2 - \sigma_2\sqrt{-2\ln \alpha}, \mu_2 + \sigma_2\sqrt{-2\ln \alpha}]$

3. D.1 Addition.

$${}^{\alpha}A_1 + {}^{\alpha}A_2 = [\{\mu_1 - \sigma_1\sqrt{-2\ln \alpha}\} + \{\mu_2 - \sigma_2\sqrt{-2\ln \alpha}\}, \{\mu_1 + \sigma_1\sqrt{-2\ln \alpha}\} + \{\mu_2 + \sigma_2\sqrt{-2\ln \alpha}\}]$$

3. D.2 Subtraction.

$${}^{\alpha}A_1 - {}^{\alpha}A_2 = [\{\mu_1 - \sigma_1\sqrt{-2\ln \alpha}\} - \{\mu_2 + \sigma_2\sqrt{-2\ln \alpha}\}, \{\mu_1 + \sigma_1\sqrt{-2\ln \alpha}\} - \{\mu_2 - \sigma_2\sqrt{-2\ln \alpha}\}]$$

3. D.3 Multiplication.

$${}^{\alpha}A_1 \times {}^{\alpha}A_2 = [\{\mu_1 - \sigma_1\sqrt{-2\ln \alpha}\} \times \{\mu_2 - \sigma_2\sqrt{-2\ln \alpha}\}, \{\mu_1 + \sigma_1\sqrt{-2\ln \alpha}\} \times \{\mu_2 + \sigma_2\sqrt{-2\ln \alpha}\}]$$

3. D.4 Division.

$$\frac{{}^{\alpha}A_1}{{}^{\alpha}A_2} = \left[\frac{\mu_1 - \sigma_1\sqrt{-2\ln \alpha}}{\mu_2 + \sigma_2\sqrt{-2\ln \alpha}}, \frac{\mu_1 + \sigma_1\sqrt{-2\ln \alpha}}{\mu_2 - \sigma_2\sqrt{-2\ln \alpha}} \right]$$

E. Cauchy and Cauchy fuzzy numbers.

Consider two Cauchy fuzzy numbers ${}^{\alpha}A_1 = \left[p_1 - q_1\sqrt{\frac{1-\alpha}{\alpha}}, p_1 + q_1\sqrt{\frac{1-\alpha}{\alpha}} \right]$ and

$${}^{\alpha}A_2 = \left[p_2 - q_2\sqrt{\frac{1-\alpha}{\alpha}}, p_2 + q_2\sqrt{\frac{1-\alpha}{\alpha}} \right].$$

3. E.1 Addition.

$${}^{\alpha}A_1 + {}^{\alpha}A_2 = \left[p_1 - q_1 \sqrt{\frac{1-\alpha}{\alpha}} + p_2 - q_2 \sqrt{\frac{1-\alpha}{\alpha}}, p_1 + q_1 \sqrt{\frac{1-\alpha}{\alpha}} + p_2 + q_2 \sqrt{\frac{1-\alpha}{\alpha}} \right]$$

3. E.2 Subtraction.

$${}^{\alpha}A_1 - {}^{\alpha}A_2 = \left[p_1 - q_1 \sqrt{\frac{1-\alpha}{\alpha}} - \left(p_2 + q_2 \sqrt{\frac{1-\alpha}{\alpha}} \right), p_1 + q_1 \sqrt{\frac{1-\alpha}{\alpha}} - \left(p_2 - q_2 \sqrt{\frac{1-\alpha}{\alpha}} \right) \right]$$

3. E.3 Multiplication.

$${}^{\alpha}A_1 \times {}^{\alpha}A_2 = \left[\left(p_1 - q_1 \sqrt{\frac{1-\alpha}{\alpha}} \right) \times \left(p_2 - q_2 \sqrt{\frac{1-\alpha}{\alpha}} \right), \left(p_1 + q_1 \sqrt{\frac{1-\alpha}{\alpha}} \right) \times \left(p_2 + q_2 \sqrt{\frac{1-\alpha}{\alpha}} \right) \right]$$

3. E.4 Division.

$$\frac{{}^{\alpha}A_1}{{}^{\alpha}A_2} = \left[\frac{\mu_1 - \sigma_1 \sqrt{-2 \ln \alpha}}{\mu_2 + \sigma_2 \sqrt{-2 \ln \alpha}}, \frac{\mu_1 + \sigma_1 \sqrt{-2 \ln \alpha}}{\mu_2 - \sigma_2 \sqrt{-2 \ln \alpha}} \right]$$

3.2 Microsoft Excel Programming for calculating the defuzzified value the Product of two fuzzy numbers:

Let $\tilde{x}_1 = [\mu_1 - \sigma_1 \sqrt{-2 \ln \alpha}, \mu_1 + \sigma_1 \sqrt{-2 \ln \alpha}]$, $\tilde{w}_1 = [\mu_2 - \sigma_2 \sqrt{-2 \ln \alpha}, \mu_2 + \sigma_2 \sqrt{-2 \ln \alpha}]$

Then $(\tilde{x}_1 \times \tilde{w}_1) = \left\{ \mu_1 - \sigma_1 \sqrt{-2 \ln \alpha} \times \mu_2 - \sigma_2 \sqrt{-2 \ln \alpha}, \mu_1 + \sigma_1 \sqrt{-2 \ln \alpha} \times \mu_2 + \sigma_2 \sqrt{-2 \ln \alpha} \right\}$

The defuzzified value of

$$\tilde{x}_1 \times \tilde{w}_1 = \frac{\mu_1 - \sigma_1 \sqrt{-2 \ln \alpha} \times \mu_2 - \sigma_2 \sqrt{-2 \ln \alpha} + \mu_1 + \sigma_1 \sqrt{-2 \ln \alpha} \times \mu_2 + \sigma_2 \sqrt{-2 \ln \alpha}}{2}$$

Now one can developed a Microsoft Excel programme to compute the value of $\tilde{x}_1 \times \tilde{w}_1$ for particular values of $\mu_1, \sigma_1, \mu_2, \sigma_2, \alpha$. For different values of α ($0 < \alpha < 1$) we get a unique value of $(\tilde{x}_1 \times \tilde{w}_1)$. Thus average defuzzified value is evaluated by taking the average value of $(\tilde{x}_1 \times \tilde{w}_1)$ for different values of α . Here we have taken α (A3) = 0.001 to 0.999 and the defuzzified values have calculated. The used algorithm of the programme in Microsoft Excel is
 ((((\$E\$2-\$F\$2*SQRT(-2*LN(A3)))*(\$G\$2-\$H\$2*SQRT(-2*LN(A3))))+(((\$E\$2+\$F\$2*SQRT(-2*LN(A3)))*(\$G\$2+\$H\$2*SQRT(-2*LN(A3)))))/2

Input values	E	F	G	H
	μ_1	σ_1	μ_2	σ_2

4. METHODOLOGY

In this section, the MCDM method proposed by D. Singh and Robert L.K. Tiong (2005) is taken for this study in which α -cut approach is used. A decision making problem is a process to finding the best option among the set of feasible alternative. A multi criteria decision making (MCDM) problem can be expressed in the matrix format as follows:

$$\begin{matrix} & C_1 & C_2 & \dots & C_m \\ A_1 & \tilde{x}_{11} & \tilde{x}_{12} & \dots & \tilde{x}_{1n} \\ A_2 & \tilde{x}_{21} & \tilde{x}_{22} & \dots & \tilde{x}_{2n} \\ \vdots & \dots & \dots & \dots & \dots \\ A_n & \tilde{x}_{m1} & \tilde{x}_{m2} & \dots & \tilde{x}_{mn} \end{matrix} \dots (i)$$

where $A_1, A_2, \dots, A_n$ are possible alternatives, $C_1, C_2, \dots, C_m$ are criteria of the alternatives and $\tilde{x}_{ij}$ is the rating of alternative A_i with respect to criteria C_j .

Linguistic variables which are used to define the weight and ratings of many qualitative criteria are expressed in terms different fuzzy numbers. In our present study positive triangular fuzzy numbers and Bell shape fuzzy

numbers are used for different weight and ratings. When the decision maker use the linguistic variables to evaluate the weight and ratings of different criteria of alternatives, the following calculations are adopted treating the decision group has K persons.

$$\tilde{x}_{ij} = \frac{1}{K} [\tilde{x}_{ij}^1 + \tilde{x}_{ij}^2 + \tilde{x}_{ij}^3 + \dots + \tilde{x}_{ij}^K], i=1,2,3,\dots,m; j=1,2,3,\dots,n \dots (ii)$$

$$\tilde{w}_j = \frac{1}{K} [\tilde{w}_j^1 + \tilde{w}_j^2 + \tilde{w}_j^3 + \dots + \tilde{w}_j^K], j=1,2,\dots,n$$

Where $\tilde{x}_{ij}^K$ and $\tilde{w}_j^K$ represent the ratings and weights of the K^{th} decision maker and $\tilde{x}_{ij}$ and $\tilde{w}_j$ represents the average ratings and average weights of the criteria respectively and m and n denotes number of alternatives and number of criteria respectively. Decision makers assign different fuzzy numbers for various criteria of alternatives. Using the ratings x_{ij} of decision makers of the alternatives

A_i with respect to criteria C_j we construct the average decision matrix $\tilde{D}$. Also the average weight matrix $\tilde{W}_j$ is

constructed by using the weights for the criterion. The average fuzzy numbers are obtained by using the above relation (ii) and also the α -cut for each average fuzzy number is calculated by using *definition 3.1*. Using the simple additive weighting method to obtain the total score for ranking order of the alternatives the following calculation is done by taking averages decision matrix and the average weight matrix.

$$\begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \dots & \tilde{x}_{1n} \\ \tilde{x}_{21} & \tilde{x}_{22} & \dots & \tilde{x}_{2n} \\ \dots & \dots & \dots & \dots \\ \tilde{x}_{m1} & \tilde{x}_{m2} & \dots & \tilde{x}_{mn} \end{bmatrix} \cdot \begin{bmatrix} \tilde{w}_1 \\ \tilde{w}_2 \\ \dots \\ \tilde{w}_n \end{bmatrix}$$

Thus total aggregated score for alternatives $A_1, A_2, \dots, A_n$

against each criterion is obtained as follows:

$$(\tilde{x}_{11} \times \tilde{w}_1 + \tilde{x}_{21} \times \tilde{w}_2 + \dots + \tilde{x}_{m1} \times \tilde{w}_n), (\tilde{x}_{12} \times \tilde{w}_1 + \tilde{x}_{22} \times \tilde{w}_2 + \dots + \tilde{x}_{m2} \times \tilde{w}_n), \dots, (\tilde{x}_{1n} \times \tilde{w}_1 + \tilde{x}_{2n} \times \tilde{w}_2 + \dots + \tilde{x}_{mn} \times \tilde{w}_n)$$

The multiplication and addition operation are performed as discussed in the definition. Their defuzzified values are obtained from *definition 3.5*. In our discussion to find defuzzified value we have taken one thousand values of α

from .001 to 1 and then their average defuzzified values have considered which gives a weight vector of defuzzified values. In our study the computation part is done by Microsoft Excel programming. On the basis of total score the order ranking of the alternatives are obtained. The methodology is verified by the following case study with five different cases.

4.1 Case Study:

Assume that university “X” desires to hire a professor for teaching fuzzy theory course. A committee of three expert decision makers, D1, D2 and D3 has been formed to conduct the interview with three eligible candidates, namely A1, A2 and A3, and to select the most suitable candidate. Five benefit criteria are considered:

(1) Publications and researches (C1), (2) Teaching skills (C2), (3) Practical experiences in industries and corporations (C3), (4) Past experiences in teaching (C4), (5) Teaching discipline (C5).

4.1.1 Case I.

In this case study weights for criteria are triangular fuzzy numbers and the ratings of alternatives with respect to each criterion are Gaussian fuzzy numbers. Decision makers choose the linguistic weighting variable (Table 1) for the criteria and the linguistic ratings variable (Table 2) to evaluate the ratings of alternatives with respect to each criterion.

Table 1: Linguistic Variable For The Importance Weight Of Each Criterion

Very Low (VL)	(0.0, 0.0, 0.1)
Low (L)	(0.0, 0.1, 0.25)
Medium Low (ML)	(0.15, 0.3, 0.45)
Medium (M)	(0.35, 0.5, 0.65)
Medium High (MH)	(0.55, 0.7, 0.85)
High (H)	(0.8, 0.9, 1.0)
Very High (VH)	(0.9, 1.0, 1.0)

Table 2: Linguistic Variables For The Ratings

Very Poor (VP)	GFN(0, 0.0065)
Poor (P)	GFN(0.2, 0.065)
Medium Poor (MP)	GFN(0.4, 0.035)
Fair (F)	GFN(0.5, 0.033)
Medium Good (MG)	GFN(0.6, 0.065)
Good (G)	GFN(0.8, 0.065)
Very Good (VG)	GFN(1, 0.0001)

Table 3: The Importance Weight Of Each Criterion Given By Decision Makers:

Decision makers Criterion	D1	D2	D3
C1	H	VH	MH
C2	VH	VH	VH
C3	VH	H	H
C4	VH	VH	VH
C5	M	MH	MH

The average weights for the criteria are calculated by using the following relation

$$\tilde{w}_j = \frac{1}{K} [\tilde{w}_j^1(+) \tilde{w}_j^2(+) \dots \dots (+) \tilde{w}_j^K]$$

$\tilde{w}_1 = (0.75, 0.87, 0.95)$, $\tilde{w}_2 = (0.9, 1.0, 1.0)$, $\tilde{w}_3 = (0.83, 0.93, 1.0)$, $\tilde{w}_4 = (0.9, 1.0, 1.0)$, $\tilde{w}_5 = (0.48, 0.63, 0.78)$.
Decision makers have given their opinion (Table 4) to get the fuzzy ratings $\tilde{x}_{ij}$ of alternative A_i under criterion C_j .

Table-4

(D1, D2, D3)	Criteria	A1	A2	A3
	C1	(MG,MG,F)	(G,MG,G)	(G,MG,MG)
	C2	(G,F,MG)	(VG,VG,G)	(VG,G,MG)
	C3	(G,G,F)	(VG,G,VG)	(G,G,MG)
	C4	(VG,MG,F)	(VG,VG,VG)	(F,MG,VG)
	C5	(F,MG,G)	(VG,G,MG)	(MG,G,G)

The average ratings of the three alternatives with respect to the criterion are calculated by using table (4) to obtain the fuzzy decision matrix $\tilde{D}$.

$$\tilde{D} = \begin{bmatrix} GFN(0.57, 0.054) & GFN(0.73, 0.065) & GFN(0.67, 0.065) \\ GFN(0.63, 0.054) & GFN(0.93, 0.022) & GFN(0.8, 0.043) \\ GFN(0.7, 0.054) & GFN(0.8, 0.043) & GFN(0.73, 0.065) \\ GFN(0.7, 0.0327) & GFN(1, 0.0001) & GFN(0.7, 0.0327) \\ GFN(0.63, 0.054) & GFN(0.8, 0.043) & GFN(0.73, 0.065) \end{bmatrix}$$

To obtain the total score for each alternative the following calculation has to be done by using the simple additive weighting method.

	A1	A2	A3	w_j
C1	$GFN(0.57, 0.054)$	$GFN(0.73, 0.065)$	$GFN(0.67, 0.065)$	$(0.75, 0.87, 0.95)$
C2	$GFN(0.63, 0.054)$	$GFN(0.93, 0.022)$	$GFN(0.8, 0.043)$	$(0.9, 1.0, 1.0)$
C3	$GFN(0.7, 0.054)$	$GFN(0.8, 0.043)$	$GFN(0.73, 0.065)$	$(0.83, 0.93, 1.0)$
C4	$GFN(0.7, 0.0327)$	$GFN(1, 0.0001)$	$GFN(0.7, 0.0327)$	$(0.9, 1.0, 1.0)$
C5	$GFN(0.63, 0.054)$	$GFN(0.8, 0.043)$	$GFN(0.73, 0.065)$	$(0.48, 0.63, 0.78)$

$j = 1, 2, 3, 4, 5$.

The α -cut value of each Gaussian fuzzy number and triangular number for $\alpha \in [0, 1]$ is obtained as follows:

$$\begin{aligned} \tilde{x}_{11} &= GFN(0.57, 0.054) = [0.57 - 0.054\sqrt{-2\ln\alpha}, 0.57 + 0.054\sqrt{-2\ln\alpha}] \\ \tilde{x}_{21} &= GFN(0.63, 0.054) = [0.63 - 0.054\sqrt{-2\ln\alpha}, 0.63 + 0.054\sqrt{-2\ln\alpha}] \\ \tilde{x}_{31} &= GFN(0.7, 0.054) = [0.7 - 0.054\sqrt{-2\ln\alpha}, 0.7 + 0.054\sqrt{-2\ln\alpha}] \\ \tilde{x}_{41} &= GFN(0.7, 0.0327) = [0.7 - 0.0327\sqrt{-2\ln\alpha}, 0.7 + 0.0327\sqrt{-2\ln\alpha}] \\ \tilde{x}_{51} &= GFN(0.63, 0.054) = [0.63 - 0.054\sqrt{-2\ln\alpha}, 0.63 + 0.054\sqrt{-2\ln\alpha}] \\ \tilde{x}_{12} &= GFN(0.73, 0.065) = [0.73 - 0.065\sqrt{-2\ln\alpha}, 0.73 + 0.065\sqrt{-2\ln\alpha}] \\ \tilde{x}_{22} &= GFN(0.93, 0.022) = [0.93 - 0.022\sqrt{-2\ln\alpha}, 0.93 + 0.022\sqrt{-2\ln\alpha}] \\ \tilde{x}_{32} &= GFN(0.8, 0.043) = [0.8 - 0.043\sqrt{-2\ln\alpha}, 0.8 + 0.043\sqrt{-2\ln\alpha}] \end{aligned}$$

$$\begin{aligned}\tilde{x}_{42} &= GFN(1, 0.0001) = \left[1 - 0.0001\sqrt{-2\ln\alpha}, 1 + 0.0001\sqrt{-2\ln\alpha}\right] \\ \tilde{x}_{52} &= GFN(0.8, 0.043) = \left[0.8 - 0.043\sqrt{-2\ln\alpha}, 0.8 + 0.043\sqrt{-2\ln\alpha}\right] \\ \tilde{x}_{13} &= GFN(0.67, 0.065) = \left[0.67 - 0.065\sqrt{-2\ln\alpha}, 0.67 + 0.065\sqrt{-2\ln\alpha}\right] \\ \tilde{x}_{23} &= GFN(0.8, 0.043) = \left[0.8 - 0.043\sqrt{-2\ln\alpha}, 0.8 + 0.043\sqrt{-2\ln\alpha}\right] \\ \tilde{x}_{33} &= GFN(0.73, 0.065) = \left[0.73 - 0.065\sqrt{-2\ln\alpha}, 0.73 + 0.065\sqrt{-2\ln\alpha}\right] \\ \tilde{x}_{43} &= GFN(0.7, 0.0327) = \left[0.7 - 0.0327\sqrt{-2\ln\alpha}, 0.7 + 0.0327\sqrt{-2\ln\alpha}\right] \\ \tilde{x}_{53} &= GFN(0.73, 0.065) = \left[0.73 - 0.065\sqrt{-2\ln\alpha}, 0.73 + 0.065\sqrt{-2\ln\alpha}\right] \\ \tilde{w}_1 &= (0.75, 0.87, 0.95) = \left[(0.87 - 0.75)\alpha + 0.75, 0.95 - (0.95 - 0.87)\alpha\right] \\ \tilde{w}_2 &= (0.9, 1, 1) = \left[(1 - 0.9)\alpha + 0.9, 1 - (1 - 1)\alpha\right] \\ \tilde{w}_3 &= (0.83, 0.93, 1) = \left[(0.93 - 0.83)\alpha + 0.83, 1 - (1 - 0.93)\alpha\right] \\ \tilde{w}_4 &= (0.9, 1, 1) = \left[(1 - 0.9)\alpha + 0.9, 1 - (1 - 1)\alpha\right] \\ \tilde{w}_5 &= (0.48, 0.63, 0.78) = \left[(0.63 - 0.48)\alpha + 0.48, 0.78 - (0.78 - 0.63)\alpha\right]\end{aligned}$$

Now, the above matrices are rewritten as follows for multiplication in simple additive weighting method.

$$\begin{bmatrix}\tilde{x}_{11} & \tilde{x}_{12} & \tilde{x}_{13} \\ \tilde{x}_{21} & \tilde{x}_{22} & \tilde{x}_{23} \\ \tilde{x}_{31} & \tilde{x}_{32} & \tilde{x}_{33} \\ \tilde{x}_{41} & \tilde{x}_{42} & \tilde{x}_{43} \\ \tilde{x}_{51} & \tilde{x}_{52} & \tilde{x}_{53}\end{bmatrix} \begin{bmatrix}\tilde{w}_1 \\ \tilde{w}_2 \\ \tilde{w}_3 \\ \tilde{w}_4 \\ \tilde{w}_5\end{bmatrix}$$

The score values for the alternatives A1, A2 and A3 on the criterion are obtained as

$$\begin{aligned}&(\tilde{x}_{11} \times \tilde{w}_1) + (\tilde{x}_{21} \times \tilde{w}_2) + (\tilde{x}_{31} \times \tilde{w}_3) + (\tilde{x}_{41} \times \tilde{w}_4) + (\tilde{x}_{51} \times \tilde{w}_5), \\ &(\tilde{x}_{12} \times \tilde{w}_1) + (\tilde{x}_{22} \times \tilde{w}_2) + (\tilde{x}_{32} \times \tilde{w}_3) + (\tilde{x}_{42} \times \tilde{w}_4) + (\tilde{x}_{52} \times \tilde{w}_5) \text{ and} \\ &(\tilde{x}_{13} \times \tilde{w}_1) + (\tilde{x}_{23} \times \tilde{w}_2) + (\tilde{x}_{33} \times \tilde{w}_3) + (\tilde{x}_{43} \times \tilde{w}_4) + (\tilde{x}_{53} \times \tilde{w}_5).\end{aligned}$$

The above calculations are done by using Microsoft Excel programming and the defuzzified values are 2.47037, 3.76211, and 3.190014. The order rankings of the alternatives are shown below:

Table 5:

Alternatives	A1	A2	A3
Score Values	2.47037	3.76211	3.190014
Rank	3 rd	1 st	2 nd

4.1.2 Case Study II.

In this case study we have taken both the weights for criteria and the ratings of alternatives with respect to each criterion are in Gaussian fuzzy numbers. Decision makers choose the linguistic weighting variable (Table 6) for the criteria and the linguistic ratings variable (Table 7) to evaluate the ratings of alternatives with respect to each criterion.

Table 6: Linguistic Variable For The Importance Weight Of Each Criterion

Very Low (VL)	GFN (0.0, .01)
Low (L)	GFN (0.3, .025)
Medium Low (ML)	GFN (0.4, 0.02)
Medium (M)	GFN (0.5, .035)
Medium High (MH)	GFN (0.7, 0.02)
High (H)	GFN (0.8, 0.04)
Very High (VH)	GFN (1, 0.001)

Table 7: Linguistic Variables For The Ratings

Very Poor (VP)	GFN(0, 0.0065)
Poor (P)	GFN(0.2, 0.065)
Medium Poor (MP)	GFN(0.4, 0.035)
Fair (F)	GFN(0.5, 0.033)
Medium Good (MG)	GFN(0.6, 0.065)
Good (G)	GFN(0.8, 0.065)
Very Good (VG)	GFN(1, 0.0001)

The average weights for the criteria are calculated by using table-3 and table-6 and to get the average fuzzy ratings $\tilde{x}_{ij}$ of alternative A_i under criterion C_j table-4 and table-7 are used and obtained as follows:

w_j		A_1	A_2	A_3
$GFN(0.83, 0.02)$	and	$C1$	$GFN(0.57, 0.054)$	$GFN(0.73, 0.065)$
$GFN(1, 0.001)$		$C2$	$GFN(0.63, 0.054)$	$GFN(0.93, 0.022)$
$GFN(0.87, 0.027)$		$C3$	$GFN(0.7, 0.054)$	$GFN(0.8, 0.043)$
$GFN(1, 0.001)$		$C4$	$GFN(0.7, 0.0327)$	$GFN(1, 0.0001)$
$GFN(0.63, 0.025)$		$C5$	$GFN(0.63, 0.054)$	$GFN(0.8, 0.043)$

The α -cut value for $\alpha \in [0, 1]$ of each Gaussian fuzzy number for both weights (w_j) for criterion and ratings

(x_{ij}) alternatives with respect to the criterion are obtained as follows:

$$\begin{aligned}\tilde{w}_1 &= GFN(0.83, 0.02) = [0.83 - 0.02\sqrt{-2\ln\alpha}, 0.83 + 0.02\sqrt{-2\ln\alpha}] \\ \tilde{w}_2 &= GFN(1, 0.001) = [1 - 0.001\sqrt{-2\ln\alpha}, 1 + 0.001\sqrt{-2\ln\alpha}] \\ \tilde{w}_3 &= GFN(0.87, 0.001) = [0.87 - 0.027\sqrt{-2\ln\alpha}, 0.87 + 0.027\sqrt{-2\ln\alpha}] \\ \tilde{w}_4 &= GFN(1, 0.001) = [1 - 0.001\sqrt{-2\ln\alpha}, 1 + 0.001\sqrt{-2\ln\alpha}] \\ \tilde{w}_5 &= GFN(0.63, 0.03) = [0.63 - 0.03\sqrt{-2\ln\alpha}, 0.63 + 0.03\sqrt{-2\ln\alpha}]\end{aligned}$$

The ratings $\tilde{x}_{ij}$ for alternatives with respect to the criterion are same as in case I. The matrices becomes

$$\begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \tilde{x}_{13} \\ \tilde{x}_{21} & \tilde{x}_{22} & \tilde{x}_{23} \\ \tilde{x}_{31} & \tilde{x}_{32} & \tilde{x}_{33} \\ \tilde{x}_{41} & \tilde{x}_{42} & \tilde{x}_{43} \\ \tilde{x}_{51} & \tilde{x}_{52} & \tilde{x}_{53} \end{bmatrix} \quad \begin{bmatrix} \tilde{w}_1 \\ \tilde{w}_2 \\ \tilde{w}_3 \\ \tilde{w}_4 \\ \tilde{w}_5 \end{bmatrix}$$

Similar multiplication and defuzzification process have performed by using Microsoft Excel programming to obtain the score values and the ranking order of the alternatives as follows:

Table 8:

Alternatives	A1	A2	A3
Score Values	2.814643	3.739678	3.158037
Rank	3 rd	1 st	2 nd

4.1.3 Case Study III.

In this case study we have taken Cauchy fuzzy numbers and Gaussian fuzzy numbers for weights for criteria and the ratings of alternatives with respect to each criterion as in the following table-9 and table-10.

Table 9: The Importance Weight Of Each Criterion

Very Low (VL)	CFN (0, 0.03)
Low (L)	CFN (0.3, 0.02)
Medium Low (ML)	CFN (0.4, 0.025)
Medium (M)	CFN (0.5, 0.02)
Medium High (MH)	CFN (0.6, 0.025)
High (H)	CFN (0.8, 0.025)
Very High (VH)	CFN (1, 0.02)

Table 10: The Ratings For Linguistic Variables

Very Poor (VP)	GFN(0, 0.0065)
Poor (P)	GFN(0.2, 0.065)
Medium Poor (MP)	GFN(0.4, 0.035)
Fair (F)	GFN(0.5, 0.033)
Medium Good (MG)	GFN(0.6, 0.065)
Good (G)	GFN(0.8, 0.065)
Very Good (VG)	GFN(1, 0.0001)

The average weights for criterion and the average ratings of alternatives with respect to each criterion are calculated by using table-3 & table-4 and table-9 & table-10 respectively.

w_j		A_1	A_2	A_3
$CFN(0.8, 0.023)$	and	C1	$GFN(0.57, 0.054)$	$GFN(0.73, 0.065)$
$CFN(1, 0.02)$		C2	$GFN(0.63, 0.054)$	$GFN(0.93, 0.022)$
$CFN(0.93, 0.015)$		C3	$GFN(0.7, 0.054)$	$GFN(0.8, 0.043)$
$CFN(1, 0.02)$		C4	$GFN(0.7, 0.0327)$	$GFN(1, 0.0001)$
$CFN(0.53, 0.023)$		C5	$GFN(0.63, 0.054)$	$GFN(0.8, 0.043)$

Now the α -cut value for $\alpha \in [0, 1]$ of each Cauchy fuzzy numbers and Gaussian fuzzy number for both weights (w_j) for criterion and ratings (x_{ij}) alternatives with respect to the criterion are obtained from definition as follows:

$$\begin{aligned}\tilde{w}_1 &= CFN(0.8, 0.023) = \left[0.8 - 0.023\sqrt{\frac{1-\alpha}{\alpha}}, 0.8 + 0.023\sqrt{\frac{1-\alpha}{\alpha}} \right] \\ \tilde{w}_2 &= CFN(1, 0.02) = \left[1 - 0.02\sqrt{\frac{1-\alpha}{\alpha}}, 1 + 0.02\sqrt{\frac{1-\alpha}{\alpha}} \right] \\ \tilde{w}_3 &= CFN(0.93, 0.015) = \left[0.93 - 0.015\sqrt{\frac{1-\alpha}{\alpha}}, 0.93 + 0.015\sqrt{\frac{1-\alpha}{\alpha}} \right] \\ \tilde{w}_4 &= CFN(1, 0.02) = \left[1 - 0.02\sqrt{\frac{1-\alpha}{\alpha}}, 1 + 0.02\sqrt{\frac{1-\alpha}{\alpha}} \right] \\ \tilde{w}_5 &= CFN(0.53, 0.023) = \left[0.53 - 0.023\sqrt{\frac{1-\alpha}{\alpha}}, 0.53 + 0.023\sqrt{\frac{1-\alpha}{\alpha}} \right]\end{aligned}$$

The ratings $\tilde{x}_{ij}$ for alternatives with respect to the criterion are same as in case I. Using the simple additive weighting method the score for the alternatives are obtained as follows:

$$\begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \tilde{x}_{13} \\ \tilde{x}_{21} & \tilde{x}_{22} & \tilde{x}_{23} \\ \tilde{x}_{31} & \tilde{x}_{32} & \tilde{x}_{33} \\ \tilde{x}_{41} & \tilde{x}_{42} & \tilde{x}_{43} \\ \tilde{x}_{51} & \tilde{x}_{52} & \tilde{x}_{53} \end{bmatrix} \quad \begin{bmatrix} \tilde{w}_1 \\ \tilde{w}_2 \\ \tilde{w}_3 \\ \tilde{w}_4 \\ \tilde{w}_5 \end{bmatrix}$$

After performing the similar calculation as earlier in case I and case II by using Microsoft Excel programming we have the following results:

Table 11:

Alternatives	A1	A2	A3
Score Values	3.361592	4.101523	3.745477
Rank	3 rd	1 st	2nd

4.1.4 Case Study IV.

In this case study we have taken both the weights for criteria and the ratings of alternatives with respect to each criterion are in Cauchy fuzzy numbers. The linguistic variables for both weights and ratings are taken in terms of Cauchy fuzzy numbers as shown below:

Table 12: The Importance Weight Of Each Criterion

Very Low (VL)	CFN (0, 0.1)
Low (L)	CFN (3, 0.2)
Medium Low (ML)	CFN (4, 0.25)
Medium (M)	CFN (5, 0.3)
Medium High (MH)	CFN (7, 0.2)
High (H)	CFN (8, 0.4)
Very High (VH)	CFN (10, 0.2)

Table 13: The Ratings For Linguistic Variables

Very Poor (VP)	CFN(0, 0.1)
Poor (P)	CFN(1, 0.2)
Medium Poor (MP)	CFN(3, 0.2)
Fair (F)	CFN(6, 0.3)
Medium Good (MG)	CFN(8, 0.3)
Good (G)	CFN(9, 0.2)
Very Good (VG)	CFN(11, 0.2)

The average weights for criterion and the average ratings of alternatives with respect to each criterion are calculated by using table-3 & table-4 and table-12 & table-13 respectively, which are shown below:

$$\begin{matrix} w_j \\ \left[\begin{matrix} CFN(8.33, 0.27) \\ CFN(10, 0.2) \\ CFN(8.67, 0.33) \\ CFN(10, 0.2) \\ CFN(6.33, 0.23) \end{matrix} \right] \end{matrix} \quad \text{and} \quad \begin{matrix} A_1 & A_2 & A_3 \\ \left[\begin{matrix} C1 & CFN(7.33, 0.3) & CFN(8.67, 0.23) & CFN(9, 0.27) \\ C2 & CFN(7.67, 0.267) & CFN(10.33, 0.2) & CFN(9.33, 0.23) \\ C3 & CFN(8, 0.23) & CFN(9.33, 0.23) & CFN(8.67, 0.23) \\ C4 & CFN(8.33, 0.27) & CFN(11, 0.2) & CFN(8.33, 0.27) \\ C5 & CFN(7.67, 0.267) & CFN(9.33, 0.23) & CFN(8.67, 0.23) \end{matrix} \right] \end{matrix}$$

The α -cut value of each Cauchy number for $\alpha \in [0,1]$ is obtained as follows:

$$\tilde{w}_1 = CFN(8.33, 0.27) = \left[(8.33 - 0.27\sqrt{\frac{1-\alpha}{\alpha}}, 8.33 + 0.27\sqrt{\frac{1-\alpha}{\alpha}} \right]$$

$$\tilde{w}_2 = CFN(10, 0.2) = \left[(10 - 0.2\sqrt{\frac{1-\alpha}{\alpha}}, 10 + 0.2\sqrt{\frac{1-\alpha}{\alpha}} \right]$$

$$\tilde{w}_3 = CFN(8.67, 0.33) = \left[(8.67 - 0.33\sqrt{\frac{1-\alpha}{\alpha}}, 8.67 + 0.33\sqrt{\frac{1-\alpha}{\alpha}} \right]$$

$$\tilde{w}_4 = CFN(10, 0.2) = \left[(10 - 0.2\sqrt{\frac{1-\alpha}{\alpha}}, 10 + 0.2\sqrt{\frac{1-\alpha}{\alpha}} \right]$$

$$\tilde{w}_5 = CFN(6.33, 0.23) = \left[(6.33 - 0.23\sqrt{\frac{1-\alpha}{\alpha}}, 6.33 + 0.23\sqrt{\frac{1-\alpha}{\alpha}} \right]$$

$$\tilde{x}_{11} = CFN(7.33, 0.3) = \left[(7.33 - 0.3\sqrt{\frac{1-\alpha}{\alpha}}, 7.33 + 0.3\sqrt{\frac{1-\alpha}{\alpha}} \right]$$

$$\tilde{x}_{21} = CFN(7.67, 0.267) = \left[(7.67 - 0.267\sqrt{\frac{1-\alpha}{\alpha}}, 7.67 + 0.267\sqrt{\frac{1-\alpha}{\alpha}} \right]$$

$$\tilde{x}_{31} = CFN(8, 0.23) = \left[(8 - 0.23\sqrt{\frac{1-\alpha}{\alpha}}, 8 + 0.23\sqrt{\frac{1-\alpha}{\alpha}} \right]$$

$$\tilde{x}_{41} = CFN(8.33, 0.27) = \left[(8.33 - 0.27\sqrt{\frac{1-\alpha}{\alpha}}, 8.33 + 0.27\sqrt{\frac{1-\alpha}{\alpha}} \right]$$

$$\tilde{x}_{51} = CFN(7.67, 0.267) = \left[(7.67 - 0.267\sqrt{\frac{1-\alpha}{\alpha}}, 7.67 + 0.267\sqrt{\frac{1-\alpha}{\alpha}} \right]$$

$$\tilde{x}_{12} = CFN(8.67, 0.23) = \left[(8.67 - 0.23\sqrt{\frac{1-\alpha}{\alpha}}, 8.67 + 0.23\sqrt{\frac{1-\alpha}{\alpha}} \right]$$

$$\tilde{x}_{22} = CFN(10.33, 0.2) = \left[(10.33 - 0.2\sqrt{\frac{1-\alpha}{\alpha}}, 10.33 + 0.2\sqrt{\frac{1-\alpha}{\alpha}} \right]$$

$$\tilde{x}_{32} = CFN(9.33, 0.23) = \left[(9.33 - 0.23\sqrt{\frac{1-\alpha}{\alpha}}, 9.33 + 0.23\sqrt{\frac{1-\alpha}{\alpha}} \right]$$

$$\tilde{x}_{42} = CFN(11, 0.2) = \left[(11 - 0.2\sqrt{\frac{1-\alpha}{\alpha}}, 11 + 0.2\sqrt{\frac{1-\alpha}{\alpha}} \right]$$

$$\tilde{x}_{52} = CFN(9.33, 0.23) = \left[(9.33 - 0.23\sqrt{\frac{1-\alpha}{\alpha}}, 9.33 + 0.23\sqrt{\frac{1-\alpha}{\alpha}} \right]$$

$$\tilde{x}_{13} = CFN(9, 0.27) = \left[(9 - 0.27\sqrt{\frac{1-\alpha}{\alpha}}, 9 + 0.27\sqrt{\frac{1-\alpha}{\alpha}} \right]$$

$$\tilde{x}_{23} = CFN(9.33, 0.23) = \left[(9.33 - 0.23\sqrt{\frac{1-\alpha}{\alpha}}, 9.33 + 0.23\sqrt{\frac{1-\alpha}{\alpha}} \right]$$

$$\tilde{x}_{33} = CFN(8.67, 0.23) = \left[(8.67 - 0.23\sqrt{\frac{1-\alpha}{\alpha}}, 8.67 + 0.23\sqrt{\frac{1-\alpha}{\alpha}} \right]$$

$$\tilde{x}_{43} = CFN(8.33, 0.27) = \left[(8.33 - 0.27\sqrt{\frac{1-\alpha}{\alpha}}, 8.33 + 0.27\sqrt{\frac{1-\alpha}{\alpha}} \right]$$

$$\tilde{X}_{53} = CFN(8.67, 0.23) = \left[(8.67 - 0.23\sqrt{\frac{1-\alpha}{\alpha}}, 8.67 + 0.23\sqrt{\frac{1-\alpha}{\alpha}} \right]$$

Now using the simple additive weighting method the score for each alternative are calculated as follows:

$$\begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} & \tilde{X}_{13} \\ \tilde{X}_{21} & \tilde{X}_{22} & \tilde{X}_{23} \\ \tilde{X}_{31} & \tilde{X}_{32} & \tilde{X}_{33} \\ \tilde{X}_{41} & \tilde{X}_{42} & \tilde{X}_{43} \\ \tilde{X}_{51} & \tilde{X}_{52} & \tilde{X}_{53} \end{bmatrix} \begin{bmatrix} \tilde{W}_1 \\ \tilde{W}_2 \\ \tilde{W}_3 \\ \tilde{W}_4 \\ \tilde{W}_5 \end{bmatrix}$$

After doing the similar calculation as in earlier cases the following results are obtained.

Table 14

Alternatives	A1	A2	A3
Score Values	340.7434	426.8025	383.1951
Rank	3 rd	1 st	2 nd

4.1.5 Case Study V.

In this case study we have taken triangular fuzzy numbers and Cauchy fuzzy numbers for the linguistic variables for weighting and the ratings of alternatives with respect to each criterion and are shown in the following tables.

Table 15: Weight Of Each Criterion

Very Low (VL)	(0.0, 0.0, 0.1)
Low (L)	(0.0, 0.1, 0.25)
Medium Low (ML)	(0.15, 0.3, 0.45)
Medium (M)	(0.35, 0.5, 0.65)
Medium High (MH)	(0.55, 0.7, 0.85)
High (H)	(0.8, 0.9, 1.0)
Very High (VH)	(0.9, 1.0, 1.0)

Table 16: The Ratings For Linguistic Variables

Very Poor (VP)	CFN(0, 0.1)
Poor (P)	CFN(1, 0.2)
Medium Poor (MP)	CFN(3, 0.2)
Fair (F)	CFN(6, 0.3)
Medium Good (MG)	CFN(8, 0.3)
Good (G)	CFN(9, 0.2)
Very Good (VG)	CFN(11, 0.2)

Thus average weights for criterion and the average ratings of alternatives with respect to each criterion are calculated by using table-3 & table-4 and table-15 & table-16 respectively, which are shown below:

w_j		A_1	A_2	A_3
$\begin{bmatrix} (0.75, 0.87, 0.95) \\ (0.9, 1.0, 1.0) \\ (0.83, 0.93, 1.0) \\ (0.9, 1.0, 1.0) \\ (0.48, 0.63, 0.78) \end{bmatrix}$	and	$\begin{bmatrix} CFN(7.33, 0.3) \\ CFN(7.67, 0.267) \\ CFN(8, 0.23) \\ CFN(8.33, 0.27) \\ CFN(7.67, 0.267) \end{bmatrix}$	$\begin{bmatrix} CFN(8.67, 0.23) \\ CFN(10.33, 0.2) \\ CFN(9.33, 0.23) \\ CFN(11, 0.2) \\ CFN(9.33, 0.23) \end{bmatrix}$	$\begin{bmatrix} CFN(9, 0.27) \\ CFN(9.33, 0.23) \\ CFN(8.67, 0.23) \\ CFN(8.33, 0.27) \\ CFN(8.67, 0.23) \end{bmatrix}$

The α -cut of the above triangular fuzzy numbers and Gaussian fuzzy numbers are calculated in the earlier cases. Thus so obtained values are again calculated for scores of the alternatives as shown follows:

$$\begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \tilde{x}_{13} \\ \tilde{x}_{21} & \tilde{x}_{22} & \tilde{x}_{23} \\ \tilde{x}_{31} & \tilde{x}_{32} & \tilde{x}_{33} \\ \tilde{x}_{41} & \tilde{x}_{42} & \tilde{x}_{43} \\ \tilde{x}_{51} & \tilde{x}_{52} & \tilde{x}_{53} \end{bmatrix} \quad \begin{bmatrix} \tilde{w}_1 \\ \tilde{w}_2 \\ \tilde{w}_3 \\ \tilde{w}_4 \\ \tilde{w}_5 \end{bmatrix}$$

After simplification by simple additive weighting method and then using defuzzification method the results follows:

Table 17:

Alternatives	A1	A2	A3
Score Values	34.2136	42.80659	38.5003
Rank	3 rd	1 st	2nd

5. CONCLUSION

In this study we have discussed D. Singh and Robert L.K. Tiong's multi criteria decision making method with α -cut approach. Various methods are developed to handle multi criteria decision making problems under fuzzy environment. Different fuzzy numbers (triangular, trapezoidal, bell shape i.e., Cauchy, Gaussian fuzzy number) are assigned for the mathematical measure of linguistic variable in MCDM methods. In this approach we can used different membership function to represents the uncertainty as the α -cut of each membership function is defined and it gives a closed interval. The methodology of α -cut approach is developed for the above mentioned method and also a case study has discussed for the selection of best alternative among three given alternatives regarding five different criteria suggested by three decision makers with the proposed method. Here we have used triangular fuzzy numbers and bell shaped membership function for the various weights and criterion. Five different case studies have considered for different combination of fuzzy numbers and have performed the arithmetic operation among the fuzzy numbers. Each case study gives the similar results. The order ranking of the alternatives is $A_2 > A_3 > A_1$. The best alternative is A_2 .

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