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The Numerical Solution of Third Order Boundary Value Problems Using Piecewise Bernstein polynomials by the Galerkin Weighted Residual Method

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Abstract: There are few numerical techniques available to solve third order boundary value problems. In this paper, we present Galerkin weighted residual method for constructing the numerical solution of third order linear and nonlinear boundary value problems with two point boundary conditions. The method is formulated as a rigorous matrix form. Several numerical examples of both linear and nonlinear boundary value problems in the literature are presented to illustrate the reliability and efficiency of the proposed method. The present method is quite efficient and yields better results when compared with the existing methods in the literature.

Keywords: Galerkin method, Third order linear and nonlinear BVPs, Bernstein Polynomials.

1. INTRODUCTION

In the literature of numerical analysis, we observe that the third order boundary value problems arise in different areas of applied mathematics and physics such as the deflection of a curved beam having a constant or a varying cross section, three layer beam, electromagnetic waves or gravity driven flows [6]. The existence and uniqueness theorem of solutions of such boundary value problems has discussed extensively in [1,2]. Caglar *et al.* [7] used fourth degree B-splines for solving third order boundary value problems. Arshad Khan and Tariq Aziz [8] numerically solved third order BVPs. using quintic splines. Loghmani and Ahmadinia [9] presented the numerical solution of third order BVPs. by the least square method with third degree B-spline function. The Sinc-collocation method is used for the numerical solution of third order BVPs in [10]. Riaz *et al.* [11] applied quartic spline solution for two point boundary problems involving third order differential equations. Moreover much attention have received for third order two point or three point BVPs. from many authors [12,13,14].

In the present paper, we shall employ the Galerkin weighted residual method [4] with Bernstein polynomials [3] as basis functions for the numerical solution of a general third order linear boundary value problem of the form:

$$c_3\frac{d^3u}{dx^3}+c_2\frac{d^2u}{dx^2}+c_1\frac{du}{dx}+c_0u=r,\ a<x<b \tag{1}$$

subject to the following boundary conditions

$$u(a)=A_0, u(b)=B_0, u'(a)=A_1 \tag{2}$$

where $A_i, i=0,1$ and $B_j, j=0$ are finite real constants and $c_i, i=0,1,\cdots 3$ and r are all continuous functions defined on the interval [a, b]. The boundary value problem (1) is solved with the boundary conditions of eqn. (2).

However, we present a short description on Bernstein polynomials in section 2. The formulation for solving linear third order BVP by the Galerkin weighted residual method with Bernstein polynomials is presented in section 3. In section 4, numerical examples for both linear and nonlinear BVPs are considered to verify the proposed formulation and the solutions are compared with the existing methods in the literature. Finally the conclusions of the paper are given in the last section.

2. BERNSTEIN POLYNOMIALS

The general form of the Bernstein polynomials [3] of nth degree over the interval $[a,b]$ is defined by

$$B_{i,n}(x)=\binom{n}{i}\frac{(x-a)^i(b-x)^{n-i}}{(b-a)^n},\qquad a\le x\le b$$

$$i=0,1,2,\ldots,n.$$

For example, the first 11 Bernstein polynomials of degree 10 over the interval $[0,1]$ are given bellow:

$B_0(x)=(1-x)^{10}$

$B_1(x)=10(1-x)^9x$

$B_2(x)=45(1-x)^8x^2$

$B_3(x)=120(1-x)^7x^3$

$B_4(x)=210(1-x)^6x^4$ $B_8(x)=45(1-x)^2x^8$

$B_5(x)=252(1-x)^5x^5$ $B_9(x)=10(1-x)x^9$

$B_6(x)=210(1-x)^4x^6$ $B_{10}(x)=x^{10}$

$B_7(x)=120(1-x)^3x^7$

Note that each of these n+1 polynomials having degree n satisfies the following properties:

(i) $B_{i,n}(x)=0$ if $i<0$ or $i>n$.

(ii) $\sum_{i=0}^{n}B_{i,n}(x)=1$

$(iii)\quad B_{i,n}(a) = B_{i,n}(b) = 0,\quad i = 1,2,\ldots,n-1$

For these properties, Bernstein polynomials are used in the trail functions satisfying the corresponding homogeneous form of the *essential* boundary conditions in the Galerkin method to solve a BVP.

3. MATRIX FORMULATION

In this section we first derived the matrix formulation for third order linear BVP and then we extend our idea for solving nonlinear BVP. To solve the boundary value problem (1) by the Galerkin weighted residual method we approximate $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} a_i\, B_{i,n}(x) \tag{3}$$

Here $\theta_0(x)$ is specified by the essential boundary conditions, $B_{i,n}(x)$ are the Bernstein polynomials which must satisfy the corresponding homogeneous boundary conditions such that $B_{i,n}(a) = B_{i,n}(b) = 0$ for each $i = 1, 2, \ldots n$.

Using eqn.(3) into eqn. (1), the Galerkin weighted residual equations are

$$\int_a^b \left[c_3 \frac{d^3\tilde{u}}{dx^3} + c_2 \frac{d^2\tilde{u}}{dx^2} + c_1 \frac{d\tilde{u}}{dx} + c_0\tilde{u} - r \right] B_{j,n}(x)dx = 0,\ j = 0,1,\cdots n \tag{4}$$

Integrating by parts the terms up to second derivative on the left hand side of (4), we get each term after applying the boundary conditions prescribed in eqn. (2) as:

$$\int_a^b c_3 \frac{d^3\tilde{u}}{dx^3} B_{j,n}(x)dx = \left[c_3 B_{j,n}(x) \frac{d^2\tilde{u}}{dx^2} \right]_a^b - \int_a^b \frac{d}{dx}\left[c_3 B_{j,n}(x)\right] \frac{d^2\tilde{u}}{dx^2} dx$$

Since $B_{j,n}(a) = B_{j,n}(b) = 0$

$$= -\left[\frac{d}{dx}\left[c_3 B_{j,n}(x)\right] \frac{d\tilde{u}}{dx} \right]_a^b + \int_a^b \frac{d^2}{dx^2}\left[c_3 B_{j,n}(x)\right] \frac{d\tilde{u}}{dx} dx \tag{5}$$

In the same way of equation (5), we have

$$\int_a^b c_2 \frac{d^2\tilde{u}}{dx^2} B_{j,n}(x)dx = -\int_a^b \frac{d}{dx}\left[c_2 B_{j,n}(x)\right] \frac{d\tilde{u}}{dx} dx \tag{6}$$

Substituting eqns.(5) and (6) into eqn. (4) and using approximation for $\tilde{u}(x)$ given in equation (3) and after rearranging the terms for the resulting equations we get a system of equations in the matrix form as

$$\sum_{i=1}^{n} D_{i,j}\, a_i = F_j,\ j = 1,2,\ldots,n \tag{7a}$$

where

$$D_{i,j} = \int_a^b \left\{ \left[\frac{d^2}{dx^2}\left[c_3 B_{j,n}(x)\right] - \frac{d}{dx}\left[c_2 B_{j,n}(x)\right] + c_1 B_{j,n}(x) \right] \frac{d}{dx}\left[B_{i,n}(x)\right] + c_0 B_{i,n}(x) B_{j,n}(x) \right\} dx$$

$$-\left[\frac{d}{dx}\left[c_3 B_{j,n}(x)\right] \frac{d}{dx}\left[B_{i,n}(x)\right] \right]_{x=b} \tag{7b}$$

$$F_j = \int_a^b \left\{ rB_{j,n}(x) + \left[-\frac{d^2}{dx^2}\left[c_3 B_{j,n}(x)\right] + \frac{d}{dx}\left[c_2 B_{j,n}(x)\right] - c_1 B_{j,n}(x) \right] \frac{d\theta_0}{dx} - c_0\theta_0 B_{j,n}(x) \right\} dx$$

$$+\left[\frac{d}{dx}\left[c_3 B_{j,n}(x)\right] \frac{d\theta_0}{dx} \right]_{x=b} - \left[\frac{d}{dx}\left[c_3 B_{j,n}(x)\right] \frac{d\theta_0}{dx} \right]_{x=a} \times A_1 \tag{7c}$$

Solving the system (7), we obtain the values of the parameters a_i and then substituting these parameters into eqn. (3), we get the approximate solution of the desired BVP (1).

For nonlinear BVP, we first compute the initial values on neglecting the nonlinear terms and using the system (7). Then using the Newton's iterative method we find the numerical approximations for desired nonlinear BVP. This formulation is described through the numerical examples in the next section.

4. NUMERICAL EXAMPLES

To test the applicability of the proposed method, we consider two linear and one nonlinear problems. For all the examples, the solutions obtained by the proposed method are compared with the exact solutions. All the calculations are performed by ***MATLAB*** **10**. The convergence of linear BVP is calculated by

$$E = \left| \tilde{u}_{n+1}(x) - \tilde{u}_n(x) \right| < \delta$$

where $\tilde{u}_n(x)$ denotes the approximate solution using *n*-th polynomials and δ depends on the problem which varies from 10^{-12}.

In addition, the convergence of nonlinear BVP is assumed when the absolute error of two consecutive iterations, δ satisfies

$$\left| \tilde{u}_n^{N+1} - \tilde{u}_n^{N} \right| < \delta$$

where N is the Newton's iteration number and δ varies from 10^{-10}.

Example 1: Consider the linear boundary value problem [8, 9, 11]

$$\frac{d^3u}{dx^3} = xu + (x^3 - 2x^2 - 5x - 3)e^x,\ 0 \le x \le 1 \tag{8a}$$

Subject to boundary conditions

$$u(0) = u(1) = 0,\ u'(0) = 1 \tag{8b}$$

The analytic solution of the above system is,

$$u(x) = x(1-x)e^x$$

In **Table 1**, we list the maximum absolute errors by applying the method illustrated in section 3. On the other hand, it is observed that the accuracy is found nearly the order

6.32×10^{-9} in [8] by A. Khan and T. Aziz and nearly the order 7.375×10^{-8} in [9] by G.B. Logmani and M. Ahmadinia and nearly the order 9.37×10^{-5} in [11] by Riaz *et al*.

Table 1: Maximum absolute errors for the example 1

x	Exact	13 Bernstein Polynomials	
		Approximate	Abs.Error
0.0	0.0000000000	0.0000000000	0.0000000E+000
0.1	0.0994653826	0.0994653826	2.3592239E-016
0.2	0.1954244413	0.1954244413	2.7755576E-016
0.3	0.2834703496	0.2834703496	5.5511151E-017
0.4	0.3580379274	0.3580379274	5.5511151E-017
0.5	0.4121803177	0.4121803177	5.5511151E-017
0.6	0.4373085121	0.4373085121	0.0000000E+000
0.7	0.4228880686	0.4228880686	0.0000000E+000
0.8	0.3560865486	0.3560865486	5.5511151E-017
0.9	0.2213642800	0.2213642800	0.0000000E+000
1.0	0.0000000000	0.0000000000	0.0000000E+000

Example 2: Consider the linear boundary value problem [5]

$$\frac{d^3u}{dx^3}+\frac{d^2u}{dx^2}+\frac{du}{dx}+u=x,\ 0\le x\le 1 \tag{9a}$$

Subject to boundary conditions

$$u(0)=1, u(1)=0,\ u'(0)=1 \tag{9b}$$

The analytic solution of the above system is,

$$u(x)=x+\frac{1}{e^x}-\sin x\,(\frac{1}{e}-1)/\sin 1-1$$

Applying the method mentioned in section 3, the maximum absolute errors are shown in **Table 2.**

Table 2: Maximum absolute errors for the example2

x	Exact	14 Bernstein Polynomials	
		Approximate	Abs. Error
0.0	1.0000000000	1.0000000000	0.0000000E+000
0.1	1.0743953317	1.0743953317	0.0000000E+000
0.2	1.0998344155	1.0998344155	0.0000000E+000
0.3	1.0798437219	1.0798437219	4.4408921E-016
0.4	1.0181389480	1.0181389480	0.0000000E+000
0.5	0.9186130796	0.9186130796	5.5511151E-016
0.6	0.7853202863	0.7853202863	0.0000000E+000
0.7	0.6224560361	0.6224560361	3.3306691E-016
0.8	0.4343338361	0.4343338361	0.0000000E+000
0.9	0.2253590182	0.2253590182	6.6613381E-016
1.0	0.0000000000	0.0000000000	0.0000000E+000

Example 3: Consider the **nonlinear** boundary value problem [8,10]

$$\frac{d^3u}{dx^3}=-2e^{-3u}+4(1+x)^{-3},\ 0\le x\le 1 \tag{10a}$$

Subject to the following boundary conditions

$$u(0)=0,\ u(1)=\ln 2,\ u'(0)=1 \tag{10b}$$

The exact solution of this BVP is $u(x)=\ln(1+x)$

Consider the approximate solution of $u(x)$ as

$$\tilde{u}(x)=\theta_0(x)+\sum_{i=1}^{n}a_i\,B_{i,n}(x),\ \ n\ge 1 \tag{11}$$

Here $\theta_0(x)=x\ln 2$ is specified by the essential boundary conditions in (10b). Also $B_{i,n}(0)=B_{i,n}(1)=0$ for each $i=1,2,\ldots,n$.

Using eqn. (11) into eqn. (10a), the Galerkin weighted residual equations are

$$\int_0^1\left[\frac{d^3\tilde{u}}{dx^3}+2e^{-3\tilde{u}}-4(1+x)^{-3}\right]B_{k,n}(x)dx=0, k=1,2,\cdots,n \tag{12}$$

Integrating first term of (12) by parts, we obtain

$$\int_0^1\frac{d^3\tilde{u}}{dx^3}B_{k,n}(x)\,dx=-\left[\frac{dB_{k,n}(x)}{dx}\frac{d\tilde{u}}{dx}\right]_0^1+\int_0^1\frac{d^2B_{k,n}(x)}{dx^2}\frac{d\tilde{u}}{dx}dx \tag{13}$$

Putting eqn. (13) into eqn. (12) and using approximation for $\tilde{u}(x)$, we obtain

$$\sum_{i=1}^{n}\left\{\int_0^1\frac{d^2B_{k,n}(x)}{dx^2}\frac{dB_{i,n}(x)}{dx}dx-\left[\frac{dB_{k,n}(x)}{dx}\frac{dB_{i,n}(x)}{dx}\right]_{x=1}\right\}a_i$$
$$=\int_0^1\left[4(1+x)^{-3}B_{k,n}(x)-\frac{d^2B_{k,n}(x)}{dx^2}\frac{d\theta_0}{dx}-2e^{-3[\theta_0+\sum_{j=1}^{n}a_jB_{j,n}(x)]}\times B_{k,n}(x)\right]dx$$
$$+\left[\frac{dB_{k,n}(x)}{dx}\frac{d\theta_0}{dx}\right]_{x=1}-\left[\frac{dB_{k,n}(x)}{dx}\frac{d\theta_0}{dx}\right]_{x=0} \tag{14}$$

The above equation (14) is equivalent to matrix form

$$DA=B+G \tag{15a}$$

where the elements of *D* and the column matrices *B* and *G* are respectively given by

$$d_{i,k}=\int_0^1\frac{d^2B_{k,n}(x)}{dx^2}\frac{dB_{i,n}(x)}{dx}dx-\left[\frac{dB_{k,n}(x)}{dx}\frac{dB_{i,n}(x)}{dx}\right]_{x=1} \tag{15b}$$

$$b_k=-2\int_0^1 e^{-3[\theta_0+\sum_{j=1}^{n}a_jB_{j,n}(x)]}\times B_{k,n}(x) \tag{15c}$$

$$g_k=\int_0^1\left[4(1+x)^{-3}B_{k,n}(x)-\frac{d^2B_{k,n}(x)}{dx^2}\frac{d\theta_0}{dx}\right]dx+\left[\frac{dB_{k,n}(x)}{dx}\frac{d\theta_0}{dx}\right]_{x=1}-\left[\frac{dB_{k,n}(x)}{dx}\frac{d\theta_0}{dx}\right]_{x=0} \tag{15d}$$

The initial values of these coefficients a_i are obtained by applying Galerkin method to the BVP neglecting the nonlinear term in (10a). That is, to find initial coefficients we will solve the system

$$DA = G \tag{16a}$$

whose matrices are constructed from

$$d_{i,k} = \int_0^1 \frac{d^2 B_{k,n}(x)}{dx^2} \frac{dB_{i,n}(x)}{dx} dx - \left[\frac{dB_{k,n}(x)}{dx} \frac{dB_{i,n}(x)}{dx} \right]_{x=1} \tag{16b}$$

$$g_k = \int_0^1 \left[4(1+x)^{-3} B_{k,n}(x) - \frac{d^2 B_{k,n}(x)}{dx^2} \frac{d\theta_0}{dx} \right] dx + \left[\frac{dB_{k,n}(x)}{dx} \frac{d\theta_0}{dx} \right]_{x=1} - \left[\frac{dB_{k,n}(x)}{dx} \frac{d\theta_0}{dx} \right]_{x=0} \tag{16c}$$

Once the initial values of a_i are obtained from eqn. (16a), they are substituted into eqn.(15a) to obtain new estimates for the values of a_i. This iteration process continues until the converged values of the unknown parameters are obtained. Substituting the final values of the parameters into eqn. (11), we obtain an approximate solution of the BVP (10).

The maximum absolute errors are shown in **Table 3** with 6 iterations. On the contrary the maximum absolute errors were found by A. Khan and T. Aziz [8] is 3.20×10^{-6} and by A. Saadatmandi and M. Razzaghi [10] is 2.0×10^{-7}

Table 3: Maximum absolute errors of example 3 using 6 iterations

x	Exact	12 Berstein Polynomials	
		Approximate	Abs. Error
0.0	0.0000000000	0.0000000000	0.000000E-000
0.1	0.0953101798	0.0953101798	2.947641×10^{-10}
0.2	0.1823215568	0.1823215568	2.084566×10^{-11}
0.3	0.2623642645	0.2623642645	4.147355×10^{-11}
0.4	0.3364722366	0.3364722366	2.208859×10^{-12}
0.5	0.4054651081	0.4054651081	2.777215×10^{-11}
0.6	0.4700036292	0.4700036292	2.647594×10^{-11}
0.7	0.5306282511	0.5306282511	1.190750×10^{-11}
0.8	0.5877866649	0.5877866649	1.816331×10^{-12}
0.9	0.6418538862	0.6418538862	7.666364×10^{-10}
1.0	0.6931471806	0.6931471806	0.000000E-000

5. CONCLUSIONS

In this paper, Galerkin method is used for finding the numerical solution of third order linear and nonlinear BVPs with Bernstein polynomials as basis functions. The numerical examples available in the literature have been considered to verify the proposed method. The study showed that our method is very convenient to solve different boundary value problems and produces reliable results. In addition, the algorithm can be coded easily and may be used for solving any higher order boundary value problems.

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