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A Theory of Lattice Interval Valued Fuzzy Sets and Fuzzy Maps Between Different Lattice Interval Valued Fuzzy Sets

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***Abstract*:** The aim of this paper is 1. to introduce the notions of, an interval valued f-set with truth values in a complete lattice of closed intervals or a simply a cloci over an arbitrary a complete lattice L, called an L-interval valued f-set or simply an L-ivf-set, an L-interval valued f-subset and to introduce an interval valued f-map between an L-interval valued f-set and an M -interval valued f-set where the complete lattice L may possibly be different from the complete lattice M, an M -interval valued f-image of an L-interval valued f-subset under an interval valued f-map and an L-interval valued f-inverse image of an M -interval valued f-subset under an interval valued f-map, and 2. to study the standard (lattice) algebraic properties of, all L-interval valued f-subsets of an L-interval valued f-set, all M -interval valued f-images of L-interval valued f-subsets under an interval valued f-map and of all L-interval valued f-inverse images of M -interval valued f-subsets under an interval valued f-map, generalizing the Theory of f-Sets.

Keywords: Fuzzy Set, Fuzzy Image, Fuzzy Inverse Image, Complete Lattice of Closed Intervals, L-Interval Valued Fuzzy Set

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I. INTRODUCTION

The traditional view in science, especially in mathematics, is to avoid uncertainty at all levels at any cost. Thus "being uncertain" is regarded as "being unscientific". But unfortunately in real life most of the information that we have to deal with is mostly uncertain.

One of the paradigm shifts in science and mathematics in this century is to accept uncertainty as part of science and the desire to be able to deal with it, as there is very little left out in the practical real world for scientific and mathematical processing without this acceptance!

One of the earliest successful attempts in this directions is the development of the Theories of Probability and Statistics. However, both of them have their own natural limitations. Another successful attempt again in the same direction is the so called Fuzzy Set Theory, introduced by Zadeh[21].

According to Zadeh[21], a fuzzy subset of a set X is any function f from the set X itself to the closed interval $[0,1]$ of real numbers. An element x belonging to the set X belongs to the fuzzy subset f with the degree of membership fx, a *real number* between 0 and 1.

Observing that fuzzy subsets themselves require a *specific* real number between/including 0 and 1 to be associated with *each* element of X, which is *not* always possible in several of the practical applications, Zadeh[22] himself introduced the so called interval valued fuzzy subsets of a set X as means to handle even *more inexact/ uncertain, but bounded* information.

Thus, an interval valued fuzzy subset of a set X is any function f from the set X itself to the complete lattice of all nonempty closed intervals of the closed interval $[0,1]$ of real numbers. An element x belonging to the set X belongs to the fuzzy subset f with the degree of membership fx, a *nonempty* closed interval in $[0,1]$.

Interestingly, in the same year 1975 that Zadeh proposed his interval valued fuzzy subsets, Grattan-Guiness[6], Jahn[7] and Sambuc[18] also proposed interval valued fuzzy subsets.

Ever since the the interval valued fuzzy subsets came into existence, once again some mathematicians started imposing and studying both algebraic and topological structures and the interested reader can refer to Biswas[1] for interval valued fuzzy subgroups; Li and Wang[8] for SH-interval-valued fuzzy subgroups and TH-interval valued fuzzy subgroups; Shaoquan[19] for interval valued fuzzy fields and for interval valued fuzzy linear spaces; Zeng-Shi[23] and Zeng-Shi-Li[24] for concepts of cut set of interval valued fuzzy subset and interval valued nested sets and for decomposition and representation theorems of interval valued fuzzy subset; Bustince[2] for interval valued fuzzy relations and applications to approximate reasoning of interval valued fuzzy subsets; Cornelis-Deschrijver-Kerre[3] for Implication in intuitionistic fuzzy subsets and interval-valued fuzzysubset theory: construction, classification, application; and Mondal-Samanta[10] for topology on interval valued fuzzy subsets.

Looking at all these and other papers in print and on-line, one thing which becomes evident is that various (lattice) algebraic properties of interval valued fuzzy images and interval valued fuzzy inverse images which, incidentally, not only play a crucial role in the study of both interval valued fuzzy algebra and interval valued fuzzy topology but also are necessary for the individual/exclusive development of Interval Valued Fuzzy Set Theory, are *not* yet studied, although these concepts of interval valued fuzzy images and interval valued fuzzy inverse images were existing since long.

Now, the aim of this paper is 1. to introduce the notions of, interval valued f-set with truth values in a complete lattice of closed intervals or a simply a **cloci**, $I^*(L)$ on a complete lattice L, called an L-interval valued f-set or simply an L-ivf-set,

an L-interval valued f-subset and

an interval valued f-map between an L-interval valued f-set and an M-interval valued f-set where the complete lattice L may possibly be different from the complete lattice M,

an M-interval valued f-image of an L-interval valued f-subset under an interval valued f-map and

an L-interval valued f-inverse image of an M-interval valued f-subset under an interval valued f-map, and

2. to study the standard (lattice) algebraic properties of,

all L-interval valued f-subsets of an L-interval valued f-set,

all M-interval valued f-images of L-interval valued f-subsets under an interval valued f-map and of all L-interval valued f-inverse images of M-interval valued f-subsets under an interval valued f-map.

Now coming back to the developments in this side of this paper, Goguen further generalized the two types of fuzzy subsets of Zadeh, namely the fuzzy subset and the interval valued f-subset, to those that take the truth values in a complete lattice. However, even though Goguen unified both of them mathematically, one must observe here that, as mentioned earlier, when it comes to practical applications, the fuzzy subsets and the interval valued f-subsets are quite different because fuzzy sets require a specific real number between 0 and 1 to be associated with *each* of its elements while interval valued f-sets require a *reasonable* interval to be associated with *each* of its elements, offering a *representation of even more uncertainty in belonging of certain elements to a set* than the fuzzy sets themselves.

Still, the following are some lacunae that one can easily observe with any of the above notions:

a. There is *no* such notion as fuzzy set (of course some mathematicians observed that one can define the notion of a fuzzy set to be the constant map assuming the value 1, but it was *not exploited* further.)
b. It is predominant in Mathematics that, for a pair of objects to be considered one as a sub object of the other, they both must be of the same type, namely, both objects are sets, both objects are pairs, both objects are triplets etc. and this *type compatibility* between set and its fuzzy subset is *absent* in the sense that fuzzy subset is a map while the set is *not*. (Of course, one can make here two arguments namely, a map is a particular type of relation which is a subset and hence a set, and thus a fuzzy subset is also a set and secondly one can identify a set with the map that takes the constant value 1; but both of them are *not* completely natural.)
c. There is *no* such notion as fuzzy map between fuzzy sets with truth values in *different* lattices
d. It is *not* possible to accommodate the notions of fuzzy weak-relative-sub algebra and fuzzy strong-relative-subalgebra in the *conventional* way
e. The Axiom of Choice is not extendable to fuzzy subsets without its dependence on the nature of the complete lattice where the fuzzy subset takes its truth values in. (Observe that the Axiom of Choice fails with the existing definitions of L-fuzzy set and L-fuzzy product as: For any pair of fuzzy sets $\overline{A},\overline{B}: X \to L$, the fuzzy product $\overline{A}\times\overline{B}$ is defined to be the fuzzy set $(\overline{A}\times\overline{B})(x) = \overline{A}x \wedge \overline{B}x$ for all $x \in X$. Letting L to be the four element diamond looking lattice with two incomparable elements α and β and letting $\overline{A}$ and $\overline{B}$ to be the constant fuzzy sets with values α and β respectively, the fuzzy product $\overline{A}\times\overline{B}$ turns out to be the empty fuzzy subset given by the constant map assuming the value 0 of L while the fuzzy subsets $\overline{A}$ and $\overline{B}$ are non-empty.
f. There is *no transparent* forgetful functor from the category of fuzzy topological spaces to the category of topological spaces which forgets the fuzzy structure.
g. There is *no transparent* forgetful functor from the category of fuzzy rings to the category of rings which forgets the fuzzy structure.
h. Last but not least, in some L-fuzzy subsets of a set, one *must* assign the value 0 for some elements of the set when actually the membership value for them is either *not* available or *not* relevant because for a fuzzy subset of a set *every* member of the set *must* be assigned a membership value.

Keeping these things in mind, Murthy[11] modified the definition of an L-fuzzy subset of a set to that of an f-set, addressing the first, second, fifth and the eighth issues above, in such a way that each f-set carries along

a) its underlying set
b) its complete lattice where the fuzzy set takes its truth values for members of its underlying set
c) its fuzzy map that specifies membership values for all elements in its underlying set and this modification resolves the above mentioned issues. Thus we have:

an f-set is a triplet $\mathsf{A} = (A, \overline{A}, L_A)$ where

(a). A is a set, called the underlying(crisp) set of A

(b). L_A is a complete lattice, called the underlying complete lattice for truth values of elements of A

(c). $\overline{A}: A \to L_A$ is a map, called the underlying fuzzy map that assigns a truth value for each element of A.

In the same paper Murthy[11] also introduced the notion of an *f-map between f-sets whose underlying complete lattices for truth values are possibly, completely different,* addressing the third issue above, along with other notions like f-image of an f-subset under an f-map and f-inverse image of an f-subset under an f-map and studied the standard (lattice) algebraic properties of, all f-subsets of an f-set, all f-images of f-subsets of an f-set under an f-map and of all f-inverse images of f-subsets of an f-set under an f-map.

For a settlement of *other* issues and for elementary studies of algebraic and topological (sub) structures on f-sets, one can refer to Murthy[13,14,15] and Murthy and Yogeswara[12].

In the present paper we generalize this Theory of f-Sets and f-Maps to Theory of Interval Valued f-Sets and Interval Valued f-Maps. Further, all counter examples in this paper can be obtained from the *corresponding* ones in the Theory of f_Sets in Murthy and Prasanna[17]. Hence *the sectional references mentioned in this paper for counter examples in the last two sections are for the above paper.*

This paper is a part of the Ph.D. Thesis for which the second author was awarded her doctoral degree in the month of August, 2012.

In Section-1, Introduction, the goal of this paper together with its lay out is described section wise.

In Section-2, Preliminaries, we recall some basic definitions and some algebraic properties in the theory Lattices Theory like poset, least and greatest elements of a poset, (least)upper bound, (greatest)lower bound, complete lattice, complete ideal, complete homomorphisms etc., were recalled along with some of their properties which are used later.

In Section-3, results about characterisation of complete ideals; complete ideals generated by a set and a union of sets, and relations between these complete ideals; lattice algebraic properties of complete ideals; lattice algebraic properties of supremums and infimums of images, inverse images and their combinations; and lattice algebraic properties of images and inverse images of ideals are recalled and several of them will be used in the last two secions.

In Section-4, results about the complete lattice of non empty closed intervals of a complete lattice; complete ideals generated by a subset and unions of subsets of the complete lattice of non empty closed intervals and relations between these complete ideals; modularity, distributivity and infinity distributivity of the complete lattice of non empty closed intervals; properties of the embedding of a complete lattice in to the complete lattice of non empty closed intervals of the same complete lattice; and complete homomorphisms between complete lattices of non empty closed intervals, are recalled and several of them will be used again in the last two secions.

In Section-5, first the notions of, L-interval valued f-set or simply L-ivf-set, L-ivf-subset of an L-ivf-set, L-ivf-union of L-ivf-subsets of an L-ivf-set, L-ivf-intersection of L-ivf-subsets of an L-ivf-set, were introduced. Then lattice algebraic properties of L-ivf-subsets of an L-ivf-set were studied.

Next, the notions of, interval valued f-maps or simply ivf-maps between L-ivf-sets of different complete lattices L, M; ivf-image of an L-ivf-subset under an ivf-map and L-ivf-inverse image of an M-ivf-subset under an ivf-map were introduced and were shown to be well defined. Later on, lattice algebraic properties of these M-ivf-images and L-ivf-inverse images of ivf-subsets under ivf-maps; and several other properties were shown to have neatly extended from f-sets and f-maps.

II. PRELIMINARIES

Some basic definitions in Lattice Theory like poset, least and greatest elements of a poset, (least) upper bound, (greatest) lower bound, complete lattice, complete ideal, complete homomorphisms etc., along with some of their properties are freely used and they can be obtained from any of the standard text books on Lattice Theory, like Szasz[20]. However some results from lattice theory are occasionally recalled for completion sake.

Note: Since IVF-Set Theory is a natural generalization of F-Set Theory, those lattice theoretic results that are developed and played an important role in the development of F-Set Theory can naturally be expected to play a similar important role even in the development of IVF-Set Theory and this is true.

Consequently, the following ***section***, namely,

Lattice theory for F-Set Theory, which appears in Murthy and Prasanna[17], will not be reproduced in this paper but ***will remain the same together with referencing in this paper as well.*** In other words, in this paper a referencing, for example, by 3.3.11(3),…………. only means that, by 3.3.11(3) of Murthy and Prasanna[17],………….. ***So, the next section begins with number 4.***

III. LATTICE THEORY OF COMPLETE LATTICES OF CLOSED INTERVALS (CLOCIS) OR COMPLETE INTERVAL-LATTICES

In this section, results about, the complete lattice of non empty closed intervals of a complete lattice; complete ideals generated by a subset and unions of subsets of the complete lattice of non empty closed intervals and relations between these complete ideals; modularity, distributivity and infinity distributivity of the complete lattice of non empty closed intervals; properties of the embedding of a complete lattice in to the complete lattice of non empty closed intervals of the same complete lattice; and complete homomorphisms between complete lattices of non empty closed intervals, are recalled from Murthy[16]. Several of these results will be used in the last two sections for the main results of this paper.

A. Complete Lattices of Closed Intervals (Clocis):

In this subsection, first a partial ordering on the collection of all non-empty closed intervals is defined with respect which it becomes a complete lattice. Then the complete ideal generated by a subset of the complete lattice of all non-empty closed intervals is obtained in terms of the left/ right end points of members of the subset. Also, every complete lattice is naturally embedded into the the complete lattice of all non-empty closed intervals in the complete lattice itself via the one-point intervals.

Later on, various properties of the map, that assigns to each subset of a given complete lattice, the subset of all nonempty closed intervals with end points in the given subset, are recalled.

Finally we see that the above map, when restricted to the complete lattice of all complete ideals in the given complete lattice, is in fact, a complete homomorphism into the the complete lattice of all complete ideals of non-empty closed intervals in the given complete lattice itself. In a counter example, we show that this restricted complete homomorphism, in fact, is not an epimorphism.

Definitions 1.1: *(a) For any complete lattice L and for any pair of elements* $\alpha,\beta \in L$ *, the subset* $\{x \in L \mid \alpha \le x \le \beta\}$ *of* L *is called the closed interval* α,β *and is denoted by* $[\alpha,\beta]$.

Clearly, for any triplet of elements $\alpha,\beta,\gamma \in L$, (1) $\alpha \le \beta$ iff $[\alpha,\beta] \ne \phi$ (2) $[\alpha,\beta]$ = $\{\gamma\}$ iff $\alpha = \beta = \gamma$ and (3) $[\alpha,\beta]$ = ϕ iff α and β are incomparable or $\beta < \alpha$.

(b) Whenever $[\alpha,\beta]$ is a non empty closed interval, for $[\alpha,\beta]$, α is called the *left end point* and β is called the *right end point*.

(c) Whenever a non empty closed interval is denoted by a single element α, its left end point is denoted by α_L and the right end point is denoted by α_R.

(d) For any $\alpha \in L$, the non empty closed interval $[\alpha,\alpha]$ = $\{\alpha\}$ is denoted by $i(\alpha)$.

(e) For any complete lattice L and for any subset A of L, the set of all non empty closed intervals with end points in A is denoted by $I^*(A)$.

Thus $I^*: \mathsf{P}(L) \to \mathsf{P}(\mathsf{P}(L))$ is a map.

(f) For any complete lattice L and for any pair of elements $\alpha,\beta \in I^*(L)$, define $\alpha \le \beta$ iff $\alpha_L \le \beta_L$ and $\alpha_R \le \beta_R$.

(g) For any subset S of $I^*(L)$, we define S_L = $\{s_L \mid s \in \mathsf{S}\} \subseteq L$ and S_R = $\{s_R \mid s \in \mathsf{S}\} \subseteq L$.

Proposition 1.2: *For any complete lattice* L *, the following are true:*

(a) for any pair of elements $\alpha,\beta \in I^*(L)$, the following are equivalent:

(a) α = β

(b) $\alpha \le \beta$, $\beta \le \alpha$ in $I^*(L)$

(c) α_L = β_L and α_R = β_R

(b) $I^*(L)$ is a complete lattice with $\le$ defined as 4.1.1(6) above.

Definition 1.3: *(a) For any complete lattice* L*, the complete lattice* $I^*(L)$ *defined as in 4.1.3(2) above is called the complete lattice of closed intervals or simply cloci or the complete interval-lattice with end points in* L.

(b) For any $S \subseteq I^*(L)$, $(\vee S)_L$ = $\vee_{s\in S}\, s_L$ = $\vee S_L$, $(\vee S)_R$ = $\vee_{s\in S}\, s_R$ = $\vee S_R$, $(\wedge S)_L$ = $\wedge_{s\in S}\, s_L$ = $\wedge S_L$ and $(\wedge S)_R$ = $\wedge_{s\in S}\, s_R$ = $\wedge S_R$ where s = $[s_L, s_R]$, S_L = $\{s_L \mid s \in S\} \subseteq L$ and S_R = $\{s_R \mid s \in S\} \subseteq L$.

In other words, $\vee S$ = $\vee_{s\in S}\, s$ = $\vee_{s\in S}[s_L,s_R]$ = $[\vee_{s\in S}\, s_L, \vee_{s\in S}\, s_R]$ = $[\vee S_L, \vee S_R]$ = $[(\vee S)_L, (\vee S)_R]$ and $\wedge S$ = $\wedge_{s\in S}\, s$ = $\wedge_{s\in S}[s_L,s_R]$ = $[\wedge_{s\in S}\, s_L, \wedge_{s\in S}\, s_R]$ = $[\wedge S_L, \wedge S_R]$ = $[(\wedge S)_L, (\wedge S)_R]$.

Since $I^*(L)$ is a complete lattice whenever L is so, the definition of complete ideal in $I^*(L)$ is the usual one in any complete lattice. However, we state it explicitly in the following for completion sake:

Definition 1.4: *For any complete lattice* L *and for any subset* J *of* $I^*(L)$, J *is a complete ideal of* $I^*(L)$ *iff (1) for all* $\phi \ne \mathsf{S} \subseteq \mathsf{J}$, $\vee \mathsf{S} \in \mathsf{J}$ *(2)* $\beta \in \mathsf{J}$, $\alpha \in I^*(L)$, $\alpha \le \beta$ *implies* $\alpha \in \mathsf{J}$ *. Clearly, the empty set is a complete ideal of* $I^*(L)$.

The following lemma will be useful when we define ivf-intersection of ivf-subsets of an ivf-set:

Lemma 1.5: *For any family of complete ideals* $(\mathsf{S}_i)_{i\in I}$ *of the complete lattice* $I^*(L)$, $\cap_{i\in I}\, \mathsf{S}_i$ *is a complete ideal of* $I^*(L)$.

Corollary 1.6: *For any complete lattice* L *and for any subset* S *of* $I^*(L)$*, the intersection of all complete ideals of* $I^*(L)$ *which contain* S*, is the unique smallest complete ideal of* $I^*(L)$ *containing* S *(cf. 2.2.2).*

Definition 1.7: *For any complete lattice* L *and forany subset* S *of* $I^*(L)$*, the unique smallest complete ideal of* $I^*(L)$ *containing the given subset* S *is called the complete ideal generated by* S *and is denoted by* $(\mathsf{S})_{I^*(L)}$ *(cf 2.2.3).*

The following lemma will be frequently used through out the development of ivf-set theory. Again it is also true in any complete lattice, in particular, in $I^*(L)$ as stated below.

Lemma 1.8: *For any complete lattice* L *and for any subset* $\phi \ne \mathsf{S} \subseteq I^*(L)$, $(\mathsf{S})_{I^*(L)}$ = $[0, \vee \mathsf{S}]$ *where* $\vee \mathsf{S}$ *is th e join of* S *in* $I^*(L)$ *. Thus* $(\mathsf{S})_{I^*(L)}$ = $\{\alpha \in I^*(L) \mid \alpha_L \le \vee_{s\in \mathsf{S}} S_L, \alpha_R \le \vee_{s\in \mathsf{S}} S_R\}$ *. However* $(\phi)_{I^*(L)}$ = ϕ.

Lemma 1.9: *For any complete lattice* L*, the inclusion map* $i: L \to I^*(L)$ *defined by* $i(s)$=$[s,s]$ *is a complete monomorphism.*

Proposition 1.10: *For any complete lattice* L, $I^*(L)$ *is a chain iff* L = $\{0,1\}$.

Lemma 1.11: *For any complete lattice* L *and for any pair of subsets* A,B *of* L, $A \subseteq B$ *iff* $I^*(A) \subseteq I^*(B)$.

Corollary 1.12: *For any pair of complete lattices* L,M *,* L = M *iff* $I^*(L)$ = $I^*(M)$.

Corollary 1.13: *For any complete lattice* L *and for any family of subsets* $(X_j)_{j\in J}$ *of* L,

(a) always $I^*(\cup_{j\in J} X_j) \supseteq \cup_{j\in J} I^*(X_j)$

(b) however equality holds whenever each X_j is a complete ideal in L.

An equality may *not* hold in (b) above if one of X_j, is *not* an ideal.

Lemma 1.14: *Let L be a complete lattice and I be a subset of L. Then the following are true for $I^*(I)$.*

(a) I is a meet (complete) semi lattice of L iff $I^*(I)$ is a meet (complete) semi lattice of $I^*(L)$.

(b) I is a join (complete) semi lattice of L iff $I^*(I)$ is a join (complete) semi lattice of $I^*(L)$.

(c) I is a (complete) sub lattice of L iff $I^*(I)$ is a (complete) sub lattice of $I^*(L)$.

(d) I is a (complete) ideal of L iff $I^*(I)$ is a (complete) ideal of $I^*(L)$.

Theorem 1.15: *For any complete lattice L and for any subset I of L, Then following are true:*

(a) I is complete infinite meet distributive sub lattice of L iff $I^*(I)$ is so of $I^*(L)$.

(b) I is complete infinite join distributive sub lattice of L iff $I^*(L)$ is so of $I^*(L)$.

(c) Consequently, I is complete infinite distributive sub lattice of L iff $I^*(I)$ is so of $I^*(L)$.

(d) I is distributive sub lattice of L iff $I^*(I)$ is distributive sub lattice of $I^*(L)$.

(e) I is modular sub lattice of L iff $I^*(I)$ is modular sub lattice of $I^*(L)$.

In 4.1.14(4), we have seen that whenever I is a complete ideal of L, $I^*(I)$ is a complete ideal of $I^*(L)$. Hence it is natural to question whether all the complete ideals of $I^*(L)$ are of the form $I^*(I)$ where I is a complete ideal of L. However, this is *not* the case and an example is given in Murthy[16].

Lemma 1.16: *The following are true in any complete lattice L:*

(a) For any $\alpha \in L$, $I^*([0,\alpha])_L$ = $[0,i\alpha]_{I^*(L)}$

(b) For any family of complete ideals $(L_{C_j})_{j\in J}$ of L, the following are true:

(i) $I^*(\vee_{j\in J} L_{C_j})$ = $\vee_{j\in J} I^*(L_{C_j})$

(ii) $I^*(\wedge_{j\in J} L_{C_j})$ = $\wedge_{j\in J} I^*(L_{C_j})$.

Lemma 1.17: *For any complete lattice L and for any subset X such that $\phi \subseteq X \subseteq L$, we have* $(I^*(X))_{I^*(L)}$ = $I^*((X)_L)$ = $(iX)_{I^*(L)}$.

Lemma 1.18: *For any complete lattice L and for any family $(X_j)_{j\in J}$ of subsets of L,*

$(\cup_{j\in J} I^*(X_j))_{I^*(L)}$ = $(I^*(\cup_{j\in J} X_j))_{I^*(L)}$ = $I^*((\cup_{j\in J} X_j)_L)$.

B. Complete Homomorphisms of Complete Lattices of Closed Intervals

In this subsection, we make a study of the complete homomorphisms of complete lattices of closed intervals induced by the underlying complete homomorphisms of complete lattices, which is essential to define and study the interval valued f-maps between an L-interval valued f-set and an M -interval valued f-set, wher the complete lattices L and M may possibly be different.

Definition 2.1: *For any pair of posets L and M and for any map $\phi: L \to M$, the map $I^*(\phi)$: $I^*(L) \to I^*(M)$,defined by $I^*(\phi)[\alpha_L, \alpha_R]$ = $[\phi\alpha_L, \phi\alpha_R]$ is called the interval map induced by ϕ*

Lemma 2.2: *For any pair of posets L, M and for any order preserving map $\phi: L \to M$, the interval map $I^*(\phi): I^*(L) \to I^*(M)$ is well defined.*

Theorem 2.3: *For any map $\phi: L \to M$ between complete lattices L and M, the interval map $I^*(\phi): I^*(L) \to I^*(M)$ is a complete homomorphism iff $\phi: L \to M$ is a complete homomorphism.*

Lemma 2.4: *For any map $\phi: L \to M$ between complete lattices L and M, the following are true:*

(a): ϕ is a monomorphism iff $I^*(\phi)$ is a monomorphism

(b): ϕ is an epimorphism iff $I^*(\phi)$ is an epimorphism

(c): ϕ is an isomorphism iff $I^*(\phi)$ is an isomorphism.

Theorem 2.5: *For any map $\phi: L \to M$ between complete lattices L and M, the following are true:*

(a): $\phi: L \to M$ is 0-preserving iff $I^*(\phi): I^*(L) \to I^*(M)$ is 0-preserving.

(b): $\phi: L \to M$ is 1-preserving iff $I^*(\phi): I^*(L) \to I^*(M)$ is 1-preserving.

(c): $\phi: L \to M$ is 0-reflecting iff $I^*(\phi): I^*(L) \to I^*(M)$ is 0-reflecting.

(d): $\phi: L \to M$ is 1-reflecting iff $I^*(\phi): I^*(L) \to I^*(M)$ is 1-reflecting.

Lemma 2.6: *For any complete homomorphism $\psi: I^*(L) \to I^*(M)$, there exists a unique complete homomorphism $\phi: L \to M$ such that ψ = $I^*(\phi)$, whenever $\Psi(i(L)) \subseteq i(M)$.*

Theorem 2.7: *For any complete homomorphism, $\eta: L \to M$ and for any $\phi \subseteq S \subseteq L$, we have*

(a) $I^*(\eta)I^*(S)$ = $I^*(\eta S)$

(b) $(I^*(\eta)I^*(S))_{I^*(M)}$ = $I^*((\eta S)_M)$ = $(I^*(\eta)(I^*(S))_{I^*(L)})_{I^*(M)}$.

Theorem 2.8: *For any complete homomorphism $\eta : L \to M$ and for any complete ideal P of M,* $I^*(\eta)^{-1}I^*(P) = I^*(\eta^{-1}P)$.

IV. L-INTERVAL VALUED FUZZY SET THEORY

In this section, first the notions of, L-interval valued f-set or simply L-ivf-set, L-ivf-subset of an L-ivf set, L-ivf-union of L-ivf subsets of an L-ivf set, L-ivf-intersection of L-ivf subsets of an L-ivf set, were introduced. Then lattice algebraic properties of L-ivf-subsets of an L-ivf-set were studied.

Next, the notions of, interval valued f-maps or simply ivf-maps between L-ivf-sets with truth values in different complete lattices of closed intervals in different complete lattices L, ivf-image of an L-ivf-subset under an ivf-map and ivf-inverse image of an M-ivf-subset under an ivf-map were introduced and were shown to be well defined. Later on, lattice algebraic properties of these ivf-images and ivf-inverse images of ivf-subsets under ivf-maps; and several other properties were shown to have neatly extended from f-sets and f-maps.

Here onwards, for convenience sake we omit L- in all the phrases L-ivf-set, L-ivf-subset, L-ivf-union, L-ivf-intersection etc..

A. L-Interval Valued Fuzzy Sets and L-Interval Valued Fuzzy Subsets:

In this subsection the notions of ivf-set, (c-total, d-total, total, strong n)-ivf-subset, ivf-union and ivf-intersection for ivf-subsets of an ivf-set are introduced.

Definition 1.1: *(a) An interval valued f-set $\boldsymbol{A}$ or simply an ivf-set is any triplet $\boldsymbol{A} = (A, \overline{A}, I^*(L_A))$, where A is a set called the underlying set for $\boldsymbol{A}$, $I^*(L_A)$ is a complete lattice of non empty closed intervals in a complete lattice L_A, called the underlying complete lattice of closed interval truth values for $\boldsymbol{A}$ and $\overline{A} : A \to I^*(L_A)$ is a map called the underlying interval valued f-map for $\boldsymbol{A}$.*

Clearly the triplet $(A, \overline{A}, I^*(L_A))$ where $A = \phi$, the empty set with no elements, $I^*(L_A) = I^*(\phi) = \phi$, the empty complete lattice of non empty closed intervals in ϕ and $\overline{A} = \phi$, the empty map, is an ivf-set, called the *empty* ivf-set.

(b) An ivf-set $\boldsymbol{A} = (A, \overline{A}, I^*(L_A))$ is it normal iff there exists an $a_0 \in A$ such that $\overline{A}a_0 = 1_{I^*(L_A)}$.

Through out this section the bold italic letters $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, \boldsymbol{E}, \boldsymbol{G}, \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ together with their suffixes always denote the ivf-sets unless otherwise stated. Also any such bold italic letter $\boldsymbol{P}$ always denotes the triplet $(P, \overline{P}, I^*(L_P))$ where P is the underlying set for $\boldsymbol{P}$, $I^*(L_P)$ is the underlying complete lattice of non empty closed intervals in the complete lattice L_P for truth values of $\boldsymbol{P}$ and $\overline{P} : P \to I^*(L_P)$ is the underlying interval valued f-map for $\boldsymbol{P}$.

Definition 1.2: *For any pairof ivf-sets $\boldsymbol{A}, \boldsymbol{B}$, $\boldsymbol{A} = \boldsymbol{B}$ iff (i) $A = B$, (ii) $I^*(L_A) = I^*(L_B)$ and (iii) $\overline{A} = \overline{B}$.*

Definitions and Statements 1.3: *Let $\boldsymbol{A}, \boldsymbol{X}$ be a pair of ivf-sets.*

(a) $\boldsymbol{A}$ is said to be an *ivf-subset* of $\boldsymbol{X}$, denoted by $\boldsymbol{A} \subseteq \boldsymbol{X}$, iff (1) $A \subseteq X$ (b) $I^*(L_A)$ is a complete ideal of $I^*(L_X)$ (3) $\overline{A} \le \overline{X} \mid A$.

(c) By 4.1.17, since $I^*(L_A)$ is a complete ideal of $I^*(L_X)$ in the above when $\boldsymbol{A}$ is an ivf-subset of $\boldsymbol{X}$, we get that L_A is a complete ideal of L_X.

(d) Clearly, the empty ivf-set is an ivf-subset of every ivf-set and for any ivf-set $\boldsymbol{X}$, the whole ivf-set $\boldsymbol{X}$ is an ivf-subset of itself.

(e) For any ivf-set $\boldsymbol{X}$, the collection of all ivf-subsets of $\boldsymbol{X}$ is denoted by IVF $(\boldsymbol{X})$

(f) $\boldsymbol{A}$ is a *d-total* ivf-subset of $\boldsymbol{X}$ iff $\boldsymbol{A}$ is an ivf-subset of $\boldsymbol{X}$ and $A = X$

(g) $\boldsymbol{A}$ is a *c-total* ivf-subset of $\boldsymbol{X}$ iff $\boldsymbol{A}$ is an ivf-subset of $\boldsymbol{X}$ and $I^*(L_A) = I^*(L_X)$

(h) $\boldsymbol{A}$ is a *total* ivf-subset of $\boldsymbol{X}$ iff $\boldsymbol{A}$ is both a c-total and a d-total ivf-subset of $\boldsymbol{X}$

(i) $\boldsymbol{A}$ is a *strong* ivf-subset of $\boldsymbol{X}$ iff $\boldsymbol{A}$ is an ivf-subset of $\boldsymbol{X}$ and $\overline{A} = \overline{X} \mid A$

(j) $\boldsymbol{A}$ is a *nivf-subset* of $\boldsymbol{X}$ iff $\boldsymbol{A}$ is the ivf-subset of $\boldsymbol{X}$ such that $\overline{A}a$ is singleton closed interval for all $a \in A$

For any family of ivf-subsets $(\boldsymbol{A}_i)_{i \in I}$ of $\boldsymbol{X}$,

(k) the *ivf-union* of $(\boldsymbol{A}_i)_{i \in I}$, denoted by $\cup_{i \in I} \boldsymbol{A}_i$, is defined by the ivf-set $\boldsymbol{A}$, where

(a) $A = \cup_{i \in I} A_i$ is the usual set union of the collection $(A_i)_{i \in I}$ of sets

(b) $I^*(L_A) = \vee_{i \in I} I^*(L_{A_i})$ where $\vee_{i \in I} I^*(L_{A_i})$ is the complete ideal generated by $\cup_{i \in I} I^*(L_{A_i})$ in $I^*(L_X)$

(c) $\overline{A} : A \to I^*(L_A)$ is defined by $\overline{A}a = \vee_{i \in I_a} \overline{A}_i a$, where $I_a = \{i \in I \mid a \in A_i\}$ and

(l) the *ivf-intersection* of $(\boldsymbol{A}_i)_{i \in I}$, denoted by $\cap_{i \in I} \boldsymbol{A}_i$, is defined by the ivf-set $\boldsymbol{A}$, where

(a) $A = \cap_{i \in I} A_i$ is the usual set intersection of the collection $(A_i)_{i \in I}$ of sets

(b) $I^*(L_A) = \cap_{i \in I} I^*(L_{A_i})$ is the usual set intersection of the complete ideals $(I^*(L_{A_i}))_{i \in I}$ in $I^*(L_X)$

(c) $\overline{A} : A \to I^*(L_A)$ by $\overline{A}a = \wedge_{i \in I} \overline{A}_i a$.

Lemma 1.4: *For any pair of ivf-sets A and B, the following are true (a) $A = B$ (b) $A \subseteq B$ and $B \subseteq A$ (c) $A = B$, $L_A = L_B$ and $\overline{A} = \overline{B}$.*

***Proof*:** (1) $(\Rightarrow)$ (2): It follows from 5.1.3. and the definition of ivf-subset.

(2) $(\Rightarrow)$ (3): $A \subseteq B$ implies $A \subseteq B$, $I^*(L_A)$ is a complete ideal of $I^*(L_B)$ and $\overline{A} \leq \overline{B} \mid A$ and $B \subseteq A$ implies $B \subseteq A$, $I^*(L_B)$ is a complete ideal of $I^*(L_A)$ and $\overline{B} \leq \overline{A} \mid B$.

Clearly from the above $A = B$, $I^*(L_A) = I^*(L_B)$ and $\overline{A} = \overline{B}$. But by 4.1.12, $I^*(L_A) = I^*(L_B)$ implies $L_A = L_B$.

(3) $(\Rightarrow)$ (1): Since $L_A = L_B$, implies $I^*(L_A) = I^*(L_B)$, clearly $A = B$.

B. *Algebra of L-Interval Valued Fuzzy Subsets:*

In this subsection some (lattice) algebraic properties of the collection of all ivf-subsets of an ivf-set are studied. Further some lattice theoretic relations between the complete lattice of all ivf-subsets of an ivf-set and the underlying complete lattice of closed intervals for truth values are established.

Lemma 2.1: *For any ivf-set $X = (X, \overline{X}, I^*(L_X))$, the following are true:*

(a) $IVF(X)$ is a complete lattice.

(b) L_X is an infinite meet distributive lattice iff $IVF(X)$ is an infinite meet distributive lattice, whenever X is a normal ivf-set.

(c) L_X is an infinite join distributive lattice iff $IVF(X)$ is an infinite join distributive lattice.

Proof: (1) First we show that $IVF(X)$ is a poset with $\leq$ defined by $B_1 \leq B_2$ iff $B_1 \subseteq B_2$ with the least element Φ and the largest element X.

From 6.1.3, it is clear that $\Phi \subseteq A \subseteq X$ for all $A \in IVF(X)$. So, $\Phi \leq A \leq X$ for all $A \in IVF(X)$ and Φ is the least element and X is the largest element in IVF($\boldsymbol{X}$).

(A): From 6.1.3, it is clear that for all $A \in IVF(X)$, $A \leq A$.

Let $B_1 \leq B_2$ and $B_2 \leq B_1$. $B_1 \leq B_2$ implies $B_1 \subseteq B_2$, $I^*(L_{B_1})$ is a complete ideal of $I^*(L_{B_2})$ and $\overline{B}_1 \leq \overline{B}_2 \mid B_1$. $B_2 \leq B_1$ implies $B_2 \subseteq B_1$, $I^*(L_{B_2})$ is a complete ideal of $I^*(L_{B_1})$ and $\overline{B}_2 \leq \overline{B}_1 \mid B_2$.

Clearly, the above implies $B_1 = B_2$, $I^*(L_{B_1}) = I^*(L_{B_2})$ and $\overline{B}_1 = \overline{B}_2$ or $B_1 = B_2$.

Lastly, let $B_1 \leq B_2$ and $B_2 \leq B_3$. $B_1 \leq B_2$ implies $B_1 \subseteq B_2$, $I^*(L_{B_1})$ is a complete ideal of $I^*(L_{B_2})$ and $\overline{B}_1 \leq \overline{B}_2 \mid B_1$. $B_2 \leq B_3$ implies $B_2 \subseteq B_3$, $I^*(L_{B_2})$ is a complete ideal of $I^*(L_{B_3})$ and $\overline{B}_2 \leq \overline{B}_3 \mid B_2$.

Clearly from the above $B_1 \subseteq B_3$, $I^*(L_{B_1})$ and $I^*(L_{B_3})$ are complete ideals of $I^*(L_X)$ such that $I^*(L_{B_1}) \subseteq I^*(L_{B_3})$ implying $I^*(L_{B_1})$ is a complete ideal of $I^*(L_{B_3})$ and $\overline{B}_1 \leq \overline{B}_3 \mid B_1$ or $B_1 \leq B_3$, implying that $IVF(X)$ is a poset.

Let $(B_j)_{j \in J}$ be a family of ivf-subsets of X.

(B): Let $B = \cup_{j \in J} B_j$. Then $B = \cup_{j \in J} B_j$, $I^*(L_B) = \vee_{j \in J} I^*(L_{B_j})$, $\overline{B}: B \to I^*(L_B)$ is defined by $\overline{B}b = \vee_{j \in J_b} \overline{B}_j b$, where $J_b = \{j \in J \mid b \in B_j\}$.

(a): Since (i) $B_j \subseteq \cup_{j \in J} B_j = B$ (ii) $I^*(L_{B_j})$ and $I^*(L_B)$ are complete ideals of $I^*(L_X)$ such that $I^*(L_{B_j}) \subseteq I^*(L_B)$, by 3.2.4(c), $I^*(L_{B_j})$ is a complete ideal of $I^*(L_B)$ (iii) for all $b \in B_j$, $\overline{B}_j b \leq \vee_{j \in J_b} \overline{B}_j b = \overline{B}b$ which implies $\overline{B}_j \leq \overline{B} \mid B_j$, we get that $B_j \subseteq B$ for all $j \in J$ or B is an upper bound for $(B_j)_{j \in J}$.

(b): Let C be an ivf-subset of X such that C is an upper bound for $(B_j)_{j \in J}$. Then B_j is an ivf-subset of C for all $j \in J$ and hence $B_j \subseteq C$, $I^*(L_{B_j})$ is a complete ideal of $I^*(L_C)$ and $\overline{B}_j \leq \overline{C} \mid B_j$.

Clearly, $B = \cup_{j \in J} B_j \subseteq C$, $I^*(L_B) = \vee_{j \in J} I^*(L_{B_j}) \subseteq I^*(L_C)$ and hence, by 3.2.4(c), $I^*(L_B)$ is a complete ideal of $I^*(L_C)$ and $\overline{B}b = \vee_{j \in J_b} \overline{B}_j b \leq \overline{C}b$ for all $b \in B$, implying that $B \subseteq C$ and that B is the least upper bound for $(B_j)_{j \in J}$ in $IVF(X)$.

(C): Let $B = \cap_{j \in J} B_j$. Then $B = \cap_{j \in J} B_j$, $I^*(L_B) = \wedge_{j \in J} I^*(L_{B_j})$ and for all $b \in B$, $\overline{B}b = \wedge_{j \in J} \overline{B}_j b$.

(a) Since (i) $B = \cap_{j \in J} B_j \subseteq B_j$ (ii) $I^*(L_B)$ and $I^*(L_{B_j})$ are complete ideals of $I^*(L_X)$ such that $I^*(L_B) \subseteq I^*(L_{B_j})$, by 3.2.4(c), $I^*(L_B)$ is a complete ideal of $I^*(L_{B_j})$ and (3) $\overline{B}b = \wedge_{j \in J} \overline{B}_j b \leq \overline{B}_j b$

for all $b \in B$ implies $\overline{B} \leq \overline{B}_j \mid B$, we get that $B \subseteq B_j$ for all $j \in J$ or B is a lower bound for $(B_j)_{j\in J}$.

(b): Let C be an ivf-subset of X such that C is a lower bound for $(B_j)_{j\in J}$. Then $C \subseteq B_j$ for all $j \in J$ and hence $B_j \supseteq C$, $I^*(L_C)$ is a complete ideal of $I^*(L_{B_j})$ and $\overline{B}_j \mid C \geq \overline{C}$.

Clearly, $B = \cap_{j\in J} B_j \supseteq C$, $I^*(L_B) = \wedge_{j\in J} I^*(L_{B_j}) \supseteq I^*(L_C)$ and hence $I^*(L_C)$ is a complete ideal of $I^*(L_B)$ and $\overline{B}b = \wedge_{j\in J_b} \overline{B}_j b \geq \overline{C}b$ for all $b \in B$, implying that $B \supseteq C$ and that B is the greatest lower bound for $(B_j)_{j\in J}$ in $IVF(X)$.

Now (A), (B) and (C) imply that $IVF(X)$ is a complete lattice.

(2): ($\Rightarrow$): Let B, C_j be ivf-subsets of X for all $j \in J$.

Let $C = \vee_{j\in J} C_j$. Then $C = \cup_{j\in J} C_j$, $I^*(L_C) = \vee_{j\in J} I^*(L_{C_j})$ and for all $c \in C$, $\overline{C}c = \vee_{j\in J_c} \overline{C}_j c$, where $J_c = \{j \in J \mid c \in C_j\}$.

Let $D = B \wedge C$. Then $D = B \cap C$, $I^*(L_D) = I^*(L_B) \cap I^*(L_C)$ and for all $d \in D$, $\overline{D}d = \overline{B}d \wedge \overline{C}d$.

Let $E_j = B \wedge C_j$. Then $E_j = B \cap C_j$, $I^*(L_{E_j}) = I^*(L_B) \cap I^*(L_{C_j})$ and for all $e \in E_j$,

$\overline{E}_j e = \overline{B}e \wedge \overline{C}_j e$.

Let $F = \vee_{j\in J} E_j$. Then $F = \cup_{j\in J} E_j$, $I^*(L_F) = \vee_{j\in J} I^*(L_{E_j})$ and for all $f \in F$, $\overline{F}f = \vee_{j\in J_f} \overline{E}_j f$, where $J_f = \{j \in J \mid f \in E_j\} = \{j \in J \mid f \in B \cap C_j\}$.

We show that $D = F$ or (a) $D = F$ (b) $I^*(L_D) = I^*(L_F)$ and (c) $\overline{D} = \overline{F}$.

(a): $D = B \cap (\cup_{j\in J} C_j) = \cup_{j\in J} (B \cap C_j) = \cup_{j\in J} E_j = F$.

(b): First by 4.1.16, $I^*(L_C) = \vee_{j\in J} I^*(L_{C_j}) = I^*(\vee_{j\in J} L_{C_j})$, $I^*(L_D) = I^*(L_B) \wedge I^*(L_C) = I^*(L_B \wedge L_C)$, $I^*(L_{E_j}) = I^*(L_B) \wedge I^*(L_{C_j}) = I^*(L_B \wedge L_{C_j})$ and $I^*(L_F) = \vee_{j\in J} I^*(L_{E_j}) = I^*(\vee_{j\in j} L_{E_j})$.

Next, by 4.1.12, the above implies, $L_C = \vee_{j\in J} L_{C_j}$, $L_D = L_B \wedge L_C$, $L_{E_j} = L_B \wedge L_{C_j}$ and $L_F = \vee_{j\in J} L_{E_j}$.

But by 3.5.2(1), $L_D = L_B \wedge L_C = L_B \wedge (\vee_{j\in J} L_{C_j}) = \vee_{j\in J} (L_B \wedge L_{C_j}) = \vee_{j\in J} L_{E_j} = L_F$.

Since $L_D = L_F$, $I^*(L_D) = I^*(L_F)$.

(c): Let $d \in D = F$. Then $\overline{D}d = \overline{B}d \wedge \overline{C}d = \overline{B}d \wedge \vee_{j\in J_d} \overline{C}_j d$, $J_d = \{j \in J \mid d \in C_j\}$ and

$\overline{F}d = \vee_{j\in J'_d} \overline{E}_j d = \vee_{j\in J'_d} (\overline{B}d \wedge \overline{C}_j d)$, $J'_d = \{j \in J \mid d \in B \cap C_j\}$.

Since (a) above implies $J_d = J'_d$ and L satisfies the infinite meet distributive law, $\overline{D}d = \overline{F}d$ or $B \wedge \vee_{j\in J} C_j = D = F = \vee_{j\in J} (B \wedge C_j)$, implying that $IVF(X)$ is an infinite meet distributive lattice.

($\Leftarrow$): Let $\alpha \in L_X$ and $(\beta_j)_{j\in J} \subseteq L_X$. Since X is a normal ivf-set, there exist an $x_0 \in X$ such that $\overline{X}x_0 = 1_{I^*(L_X)}$. For any $\alpha \in L_X$, define $A_\alpha = (X, \overline{A}_\alpha, I^*(L_X))$ where $\overline{A}_\alpha : X \to I^*(L_X)$ is defined by $\overline{A}_\alpha x_0 = i\alpha$ and $\overline{A}_\alpha x = 0_{I^*(L_X)}$ for $x \neq x_0$. Then A_α is an ivf-subset of X for all $\alpha \in L_A$ because $\overline{X}x_0 = 1 \geq i\alpha = A_\alpha x_0$.

Let $D = A_\alpha \wedge (\vee_{j\in J} A_{\beta_J})$ and $E = \vee_{j\in J} (A_\alpha \wedge A_{\beta_J})$. Then $IVF(X)$ is infinite meet distributive lattice and so $D = E$ and in particular $\overline{D} = \overline{E}$. Clearly, by the definition of A_α, since $i : L_X \to I^*(L_X)$ is a complete monomorphism, $\overline{D}x_0 = i\alpha \wedge (\vee_{j\in J} i\beta_j) = i\alpha \wedge i(\vee_{j\in J} \beta_j) = i(\alpha \wedge \vee_{j\in J} \beta_j)$ and

$\overline{E}x_0 = \vee_{j\in J} (i\alpha \wedge i\beta_j) = \vee_{j\in J} i(\alpha \wedge \beta_j) = i(\vee_{j\in J} (\alpha \wedge \beta_j))$.

Now $\overline{D}x_0 = \overline{E}x_0$ implies $\alpha \wedge \vee_{j\in J} \beta_j = \vee_{j\in J} (\alpha \wedge \beta_j)$.

(3)($\Rightarrow$): Let B, C_j be ivf-subsets of X for all $j \in J$.

Let $C = \wedge_{j\in J} C_j$. Then $C = \cap_{j\in J} C_j$, $I^*(L_C) = \wedge_{j\in J} I^*(L_{C_j})$ and for all $c \in C$, $\overline{C}c = \wedge_{j\in J_c} \overline{C}_j c$, where $J_c = \{j \in J \mid b \in c_j\}$.

Let $D = B \vee C$. Then $D = B \cup C$, $I^*(L_D) = I^*(L_B) \vee I^*(L_C)$ and for all $d \in D$, $\overline{D}d = \overline{B}d \vee \overline{C}d$.

Let $E_j = B \vee C_j$. Then $E_j = B \cup C_j$, $I^*(L_{E_j}) = I^*(L_B) \vee I^*(L_{C_j})$ and

for all $e \in E_j$, $\overline{E}_j e = \overline{B}e \vee \overline{C}_j e$.

Let $F = \wedge_{j\in J} E_j$. Then $F = \cap_{j\in J} E_j$, $I^*(L_F) = \wedge_{j\in J} I^*(L_{E_j})$ and for all $f \in F$, $\overline{F}f = \wedge_{j\in J_f} \overline{E}_j f$, where $J_f = \{j \in J \mid f \in E_j\} = \{j \in J \mid f \in B \cup C_j\}$.

We show that $D = F$ or (a) $D = F$ (b) $I^*(L_D) = I^*(L_F)$ and (c) $\overline{D} = \overline{F}$.

(a): $D = B \cup (\cap_{j\in J} C_j) = \cap_{j\in J} (B \cup C_j) = \cap_{j\in J} E_j = F$.

(b): First by 4.1.16, $I^*(L_C) = \wedge_{j\in J} I^*(L_{C_j}) = I^*(\wedge_{j\in J} L_{C_j})$, $I^*(L_D) = I^*(L_B) \vee I^*(L_C) = I^*(L_B \vee L_C)$, $I^*(L_{E_j}) = I^*(L_B) \vee I^*(L_{C_j}) = I^*(L_B \vee L_{C_j})$ and $I^*(L_F) = \wedge_{j\in J} I^*(L_{E_j}) = I^*(\wedge_{j\in j} L_{E_j})$.

Next, by 4.1.12, the above implies, $L_C = \wedge_{j\in J} L_{C_j}$, $L_D = L_B \vee L_C$, $L_{E_j} = L_B \vee L_{C_j}$ and $L_F = \wedge_{j\in J} L_{E_j}$.

But by 3.5.2(2), $L_D = L_B \vee L_C = L_B \vee (\wedge_{j\in J} L_{C_j}) = \wedge_{j\in J} (L_B \vee L_{C_j}) = \wedge_{j\in J} L_{E_j} = L_F$.

Since $L_D = L_F$, $I^*(L_D) = I^*(L_F)$.

(c): Let $d \in D = F$. Then $\overline{D}d = \overline{B}d \vee \overline{C}d = \overline{B}d \vee \wedge_{j\in J_d} \overline{C}_j d$, $J_d = \{j \in J \mid d \in C_j\}$ and $\overline{F}d = \wedge_{j\in J'_d} \overline{E}_j d = \wedge_{j\in J'_d} (\overline{B}d \vee \overline{C}_j d)$, $J'_d = \{j \in J \mid d \in B \cup C_j\}$.

Since (a) implies $J_d = J'_d$ and L satisfies the infinite join distributive law, $\overline{D}d = \overline{F}d$ or $B \vee \wedge_{j\in J} C_j = D = F = \wedge_{j\in J} (B \vee C_j)$, implying that $IVF(X)$ is an infinite join distributive law.

($\Leftarrow$): Let $\alpha \in L_X$ and $(\beta_j)_{j\in J} \subseteq L_X$. Since X is a normal ivf-set, there exist an $x_0 \in X$ such that $\overline{X}x_0 = 1_{I^*(L_X)}$. For any $\alpha \in L_X$, define $A_\alpha = (X, \overline{A}_\alpha, I^*(L_X))$ where $\overline{A}_\alpha : X \to I^*(L_X)$ is defined by $\overline{A}_\alpha x_0 = i\alpha$ and $\overline{A}_\alpha x = 0_{I^*(L_X)}$ for $x \neq x_0$. Then A_α is an ivf-subset of X for all $\alpha \in L_A$ because $\overline{X}x_0 = 1 \geq \alpha = A_\alpha x_0$.

Let $D = A_\alpha \vee (\wedge_{j\in J} A_{\beta_J})$ and $E = \wedge_{j\in J} (A_\alpha \vee A_{\beta_J})$. Then $IVF(X)$ is an infinite join distributive lattice and so $D = E$ and in particular $\overline{D} = \overline{E}$. Clearly, by the definition of A_α, $\overline{D}x_0 = i\alpha \vee (\wedge_{j\in J} i\beta_j) = i\alpha \vee i(\wedge_{j\in J} \beta_j) = i(\alpha \vee \wedge_{j\in J} \beta_j)$ and $\overline{E}x_0 = \wedge_{j\in J} (i\alpha \vee i\beta_j) = \wedge_{j\in J} i(\alpha \vee \beta_j) = i(\wedge_{j\in J} (\alpha \vee \beta_j))$.

Now $\overline{D}x_0 = \overline{E}x_0$ implies $\alpha \vee \wedge_{j\in J} \beta_j = \wedge_{j\in J} (\alpha \vee \beta_j)$.

C. Fuzzy Maps Between An L-Interval Valued Fuzzy Set and An M-Interval Valued Fuzzy Set:

In this subsection the notions of, an (increasing, decreasing, preserving) interval valued f-map or simply an ivf-map between an L-ivf-set and an M-ivf-set and the ivf-composition of such ivf-maps were introduced.

Definition 3.1: *A generalised ivf-map from A to B is any pair (f, ψ), denoted by $(f, \psi): A \to B$, where $f : A \to B$ is a set map and $\psi : I^*(L_A) \to I^*(L_B)$ is a complete homomorphism.*

Definition 3.2: *An ivf-map from A to B is any pair $F = (f, I^*(L_f))$, denoted by $F : A \to B$, where $f : A \to B$ is a set map and $L_f : L_A \to L_B$ is a complete homomorphism.*

Definition 3.3: *For any ivf-map $(f, I^*(L_f)) : (A, \overline{A}, I^*(L_A)) \to (B, \overline{B}, I^*(L_B))$,*

(i) $(f, I^*(L_f))$ is *increasing*, denoted by F_i, iff $\overline{B}f \geq I^*(L_f)\overline{A}$

(ii) $(f, I^*(L_f))$ is *decreasing*, denoted by F_d, iff $\overline{B}f \leq I^*(L_f)\overline{A}$

(iii) $(f, I^*(L_f))$ is *preserving*, denoted by F_p, iff $\overline{B}f = I^*(L_f)\overline{A}$

Definition 3.4: *For any pair of ivf-maps $F = (f, I^*(L_f)) : A \to B$ and $G = (g, I^*(L_g)) : B \to C$, the ivf-composition of F by G, denoted by $GF : A \to C$, is defined by the ivf-map* $GF = (gf, I^*(L_g)I^*(L_f))$.

D. M-Interval Valued Fuzzy Images and L-Interval Valued Fuzzy Inverse Images of Interval Valued Fuzzy Subsets:

In this subsection the notions of, the M-ivf-image of an L-ivf-subset under an ivf-map and the L-ivf-inverse image of an M-ivf-subset under an ivf-map were introduced and were shown to be well defined.

Lemma 4.1: *For any ivf-map* $(f, I^*(L_f))$: $(A, \overline{A}, I^*(L_A)) \to (B, \overline{B}, I^*(L_B))$, *the following are true:*

(a) For any ivf-subset $(C, \overline{C}, I^*(L_C))$ of $(A, \overline{A}, I^*(L_A))$, the ivf-set D where D = fC , $I^*(L_D)$ = $(I^*(L_f)I^*(L_C))_{I^*(L_B)}$ and $\overline{D}: D \to I^*(L_D)$ is defined by

$\overline{D}d = \overline{B}d \wedge \vee I^*(L_f)\overline{C}(f^{-1}d \cap C)$ for all $d \in D$, is an ivf-subset of B.

(b) For any ivf-subset $(D, \overline{D}, I^*(L_D))$ of $(B, \overline{B}, I^*(L_B))$, the ivf-set C where C = $f^{-1}D$, $I^*(L_C)$ = $I^*(L_f)^{-1}I^*(L_D)$ and $\overline{C}: C \to I^*(L_C)$ is defined by $\overline{C}c = \overline{A}c \wedge \vee I^*(L_f)^{-1}\overline{D}fc$ for all $c \in C$, is an ivf-subset of A.

***Proof*:** (a) Since $C \subseteq A$, $C \subseteq A$, $I^*(L_C)$ is a complete ideal of $I^*(L_A)$ and $\overline{C} \le \overline{A} | C$.

Therefore, D = $fC \subseteq fA \subseteq B$ and $I^*(L_D)$ = $(I^*(L_f)I^*(L_C))_{I^*(L_B)}$ is a complete ideal of $I^*(L_B)$.

Further, since $f^{-1}d \cap C \subseteq C$, we have $\overline{C}(f^{-1}d \cap C) \subseteq \overline{C}C \subseteq I^*(L_C)$. So, $I^*(L_f)\overline{C}(f^{-1}d \cap C) \subseteq I^*(L_f)I^*(L_C)$ $\subseteq$ $(I^*(L_f)I^*(L_C))_{I^*(L_B)}$ = $I^*(L_D)$.

Now since $I^*(L_D)$ is a complete ideal, we get that $\vee I^*(L_f)\overline{C}(f^{-1}d \cap C) \in I^*(L_D)$ and $\overline{D}d$ = $\overline{B}d \wedge \vee I^*(L_f)\overline{C}(f^{-1}d \cap C) \in I^*(L_D)$.

Thus the ivf-image of an ivf-subset is a well-defined ivf-subset of B.

(b) Since $D \subseteq B$, $D \subseteq B$, $I^*(L_D)$ is a complete ideal of $I^*(L_B)$ and $\overline{D} \le \overline{B} | D$.

Therefore C = $f^{-1}D \subseteq f^{-1}B \subseteq A$. Further since the inverse image of a complete ideal under a complete homomorphism is a complete ideal, $I^*(L_C)$ = $I^*(L_f)^{-1}I^*(L_D)$ is a complete ideal of $I^*(L_A)$. Also C = $f^{-1}D$ implies fC = D which in turn implies $\overline{D}fc \in \overline{D}D \subseteq I^*(L_D)$.

Therefore $I^*(L_f)^{-1}\overline{D}fc \subseteq I^*(L_f)^{-1}I^*(L_D)$ = $I^*(L_C)$.

Now, since $I^*(L_C)$ is a complete ideal, we get that $\vee I^*(L_f)^{-1}\overline{D}fc \in I^*(L_C)$ and hence $\overline{C}c$ = $\overline{A}c \wedge \vee I^*(L_f)^{-1}\overline{D}fc \in I^*(L_C)$, implying that the ivf-inverse image of an ivf-subset is a well-defined ivf-subset of A.

Definition 4.2: *Let* $F: A \to B$ *be an ivf-map. Then*

(a) For any ivf-subset C of A, the *ivf-image* of C under F , denoted by FC, is defined by D, where (a) D = fC (b) $I^*(L_D)$ = $(I^*(L_f)I^*(L_C))_{I^*(L_B)}$ and (c) $\overline{D}d$ = $\overline{B}d \wedge \vee I^*(L_f)\overline{C}(f^{-1}d \cap C)$ for all $d \in D$.

(b) For any ivf-subset D of B, the *ivf-inverse image* of D under F, denoted by $F^{-1}D$, is defined by C, where (a) C = $f^{-1}D$ (b) $I^*(L_C)$ = $I^*(L_f)^{-1}I^*(L_D)$ and (c) $\overline{C}c$ = $\overline{A}c \wedge \vee I^*(L_f)^{-1}\overline{D}fc$ for all $c \in C$.

E. Properties of M-Interval Valued Fuzzy Images and L-Interval Valued Fuzzy Inverse Images:

In this subsection some standard lattice algebraic properties of the collections of, M -ivf-images of L -ivf-subsets under an ivf-map and the L -ivf-inverse images of M -ivf-subsets under an ivf-map are studied in detail.

Further, all counter examples in this subsection can be obtained from the *corresponding* ones in the Theory of f_Sets And f-Maps-Revisited in Murthy and Prasanna[17]. Hence *the sectional references mentioned in this section for counter examples are for the above paper. Also, as mentioned earlier in a Note before Section 4, a referencing, for example, by 3.3.11(3),……… only means that, by 3.3.11(3) of Murthy and Prasanna[17], ……….*

Definitions 5.1: *(a) Let* $F: A \to B$ *be an ivf-map and* $C \subseteq B$. *Then* C *is said to be an* $I^*(L_f)$ *-regular ivf-subset of* B *iff* $I^*(L_C) \subseteq I^*(L_f)I^*(L_A)$.

(b) An f-map $F = (f, I^*(L_f))$ is

(a) *0-preserving*, or simply 0-p iff $I^*(L_f)$ is a 0-preserving complete homomorphism (Cf. 3.3.6)

(b) *1-preserving* or simply 1-p iff $I^*(L_f)$ is a 1-preserving complete homomorphism (Cf.3.3.6)

(c) *0-reflecting* or simply 0-r iff $I^*(L_f)$ is a 0-reflecting complete homomorphism (Cf.3.3.18) and

(d) *1-reflecting* or simply 1-r iff $I^*(L_f)$ is a 1-reflecting complete homomorphism (Cf.3.3.18).

Proposition 5.2: *for any ivf-map* $F: A \to B$ *and for any pair of ivf-subsets* A_1 *and* A_2 *of* A *such that* $A_1 \subseteq A_2$, *we always have* $F_* A_1 \subseteq F_* A_2$ *whenever* $*$ = i *or* d *or* p .

Proof: Let $D_1 = FA_1$ .Then $D_1 = fA_1$, $I^*(L_{D_1})$ = $(I^*(L_f)I^*(L_{A_1}))_{I^*(L_B)}$ and

$\overline{D}_1 d = \overline{B}d \wedge \vee I^*(L_f)\overline{A}_1(f^{-1}d \cap A_1)$ for all $d \in D_1$.

Let $D_2 = FA_2$. Then $D_2 = fA_2$, $I^*(L_{D_2})$ = $(I^*(L_f)I^*(L_{A_2}))_{I^*(L_B)}$ and

$\overline{D}_2 d = \overline{B}d \wedge \vee I^*(L_f)\overline{A}_2(f^{-1}d \cap A_2)$ for all $d \in D_2$.

We show that $D_1 \subseteq D_2$ or (a) $D_1 \subseteq D_2$ (b) $I^*(L_{D_1})$ is a complete ideal of $I^*(L_{D_2})$ and (c) $\overline{D}_1 \le \overline{D}_2 \mid D_1$.

Since $A_1 \subseteq A_2$, we have $A_1 \subseteq A_2$, $I^*(L_{A_1})$ is a complete ideal of $I^*(L_{A_2})$ and $\overline{A}_1 \le \overline{A}_2 \mid A_1$.

(a): $D_1 = fA_1 \subseteq fA_2 = D_2$, since $A_1 \subseteq A_2$.

(b): Since $I^*(L_{A_1}) \subseteq I^*(L_{A_2})$, we have $I^*(L_f)I^*(L_{A_1}) \subseteq I^*(L_f)I^*(L_{A_2})$ and so $I^*(L_{D_1})$ = $(I^*(L_f)I^*(L_{A_1}))_{I^*(L_B)}$ is a complete ideal of $(I^*(L_f)I^*(L_{A_2}))_{I^*(L_B)} = I^*(L_{D_2})$, by 3.2.3(7).

(c): Let $d \in D_1$. Since $f^{-1}d \cap A_1 \subseteq f^{-1}d \cap A_2$ and $\overline{A}_1 \le \overline{A}_2 \mid A_1$, we get that $I^*(L_f)\overline{A}_1 \le I^*(L_f)\overline{A}_2 \mid A_1$

By 3.4.8, $\vee I^*(L_f)\overline{A}_1(f^{-1}d \cap A_1) \le \vee I^*(L_f)\overline{A}_2$ $(f^{-1}d \cap A_1) \le \vee I^*(L_f)\overline{A}_2$ $(f^{-1}d \cap A_2)$ which now implies $\overline{D}_1 d$ = $\overline{B}d \wedge \vee I^*(L_f)\overline{A}_1$ $(f^{-1}d \cap A_1) \le$ $\overline{B}d \wedge \vee I^*(L_f)\overline{A}_2$ $(f^{-1}d \cap A_2)$ = $\overline{D}_2 d$ or $\overline{D}_1 \le \overline{D}_2 \mid D_1$.

Proposition 5.3: *For any ivf-map $F: A \to B$ and for any pair of ivf-subsets B_1 and B_2 of B such that $B_1 \subseteq B_2$ and B_2 is $I^*(L_f)$ -regular, we have $F_*^{-1}B_1 \subseteq F_*^{-1}B_2$ whenever $* = i$ or d or p.*

Proof :Let $F^{-1}B_1 = A_1$. Then $A_1 = f^{-1}B_1$, $I^*(L_{A_1})$ = $I^*(L_f)^{-1}I^*(L_{B_1})$ and

$\overline{A}_1 a = \overline{A}a \wedge \vee I^*(L_f)^{-1}\overline{B}_1 fa$ for all $a \in A_1$.

Let $F^{-1}B_2 = A_2$. Then $A_2 = f^{-1}B_2$, $I^*(L_{A_2})$ = $I^*(L_f)^{-1}I^*(L_{B_2})$ and $\overline{A}_2 a = \overline{A}a \wedge \vee I^*(L_f)^{-1}\overline{B}_2 fa$ for all $a \in A_2$.

We show that $A_1 \subseteq A_2$ or (a) $A_1 \subseteq A_2$ (b) $I^*(L_{A_1})$ is a complete ideal of $I^*(L_{A_2})$ and (c) $\overline{A}_1 \le \overline{A}_2 \mid A_1$.

Since $B_1 \subseteq B_2$, we have $B_1 \subseteq B_2$, $I^*(L_{B_1})$ is a complete ideal of $I^*(L_{B_2})$ and $\overline{B}_1 \le \overline{B}_2 \mid B_1$.

(a):Since $B_1 \subseteq B_2$,we have $A_1 = f^{-1}B_1 \subseteq f^{-1}B_2 = A_2$.

(b): Since $I^*(L_{B_1}) \subseteq I^*(L_{B_2})$, we have $I^*(L_{A_1}) = I^*(L_f^{-1})I^*(L_{B_1}) \subseteq I^*(L_f^{-1})I^*(L_{B_2}) = I^*(L_{A_2})$

So, by 3.2.4(c), $I^*(L_{A_1})$ is a complete ideal of $I^*(L_{A_2})$.

(c): Let $a \in A_1 = f^{-1}B_1$ be fixed. Then $fa \in B_1 \subseteq B_2$, $\overline{A}_1 a = \overline{A}a \wedge \vee I^*(L_f)^{-1}\overline{B}_1 fa$ and $\overline{A}_2 a = \overline{A}a \wedge \vee I^*(L_f)^{-1}\overline{B}_2 fa$.

Therefore it is enough to show that $\vee I^*(L_f)^{-1}\overline{B}_1 fa \le \vee I^*(L_f)^{-1}\overline{B}_2 fa$.

Since $a \in A = f^{-1}B_1$ and $\overline{B}_1 \le \overline{B}_2 \mid B_1$, we have $fa \in B_1 \subseteq B_2$ and $\overline{B}_1 fa \le \overline{B}_2 fa$.

Since $\overline{B}_2 fa \in I^*(L_f)I^*(L_A)$, by $I^*(L_f)$ -regularity of B_2 and by join monotonicity of $I^*(L_f)^{-1}$ as in 3.3.2, we get that $\vee I^*(L_f)^{-1}\overline{B}_1 fa \le \vee I^*(L_f)^{-1}$ $\overline{B}_2 fa$, as required.

The above proposition is *not* true if B_2 is *not* $I^*(L_f)$ -regular and the Example 4.5.7 serves here also.

Proposition 5.4: *For any o-p ivf-map $F: A \to B$ and for any ivf-subset C of A, $C \subseteq F_*^{-1}F_*C$ whenever $*$ = i or p.*

Proof: Let $FC = D$ .Then $D = fC, I^*(L_D) = (I^*(L_f)I^*(L_C))_{I^*(L_B)}$ and $\overline{D}d = \overline{B}d \wedge \vee I^*(L_f)\overline{C}(f^{-1}d \cap C)$ for all $d \in D$.

Let $F^{-1}D = E$. Then $E = f^{-1}D, I^*(L_E) = I^*(L_f)^{-1}I^*(L_D)$ and $\overline{E}e = \overline{A}e \wedge \vee I^*(L_f)^{-1}\overline{D}fe$ for all $e \in E$.

We show that $C \subseteq E$ or (a) $C \subseteq E$ (b) $I^*(L_C)$ is a complete ideal of $I^*(L_E)$ (c) $\overline{C} \le \overline{E} \mid C$.

(a): $C \subseteq f^{-1}fC = f^{-1}D = E$.

(b): $I^*(L_C) \subseteq I^*(L_f)^{-1}$ $I^*(L_f)$ $I^*(L_C) \subseteq$ $I^*(L_f)^{-1}$ $(I^*(L_f)$ $I^*(L_C))_{I^*(L_B)} = I^*(L_f)^{-1}I^*(L_D) =$ $I^*(L_E)$ Since $I^*(L_C)$ and $I^*(L_E)$ are complete ideals of $I^*(L_A)$ such that $I^*(L_C) \subseteq I^*(L_E)$,we get that $I^*(L_C)$ is a complete ideal of $I^*(L_E)$ by 3.2.4(c).

(c): Let $c \in C$ be fixed. Then $\overline{E}c = \overline{A}c \wedge \vee I^*(L_f)^{-1}\overline{D}fc$, where $\overline{D}fc = \overline{B}fc \wedge \vee I^*(L_f)\overline{C}(f^{-1}fc \cap C)$ = $\overline{B}fc \wedge \vee_{a \in f^{-1}fc \cap C} I^*(L_f)\overline{C}a$.

Since $I^*(L_f)$ is increasing, $\overline{B}fc \geq I^*(L_f)\overline{A}c$. But $I^*(L_f)\overline{A}c \geq I^*(L_f)\overline{C}c$ because $\overline{A}|C \geq \overline{C}$ and $c \in C$. Further, for all $a \in f^{-1}fc \cap C$, $fa = fc$ and $\overline{B}fa = \overline{B}fc$. So, from the above $\overline{B}fc = \overline{B}fa \geq I^*(L_f)\overline{A}a \geq I^*(L_f)\overline{C}a$ for all $a \in f^{-1}fc \cap C$, implying $\overline{B}fc \geq \vee I^*(L_f)\overline{C}(f^{-1}fc \cap C)$. Therefore, $\overline{D}fc$ = $\overline{B}fc \wedge \vee I^*(L_f)\overline{C}(f^{-1}fc \cap C)$ = $\vee I^*(L_f)\overline{C}(f^{-1}fc \cap C)$.

But $\overline{D}fc$ = $\vee I^*(L_f)\overline{C}(f^{-1}fc \cap C)$ = $I^*(L_f)(\vee\overline{C}(f^{-1}fc \cap C))$ because $f^{-1}fc \cap C \neq \phi$ and hence $\overline{C}(f^{-1}fc \cap C) \neq \phi$ and $I^*(L_f)$ is a complete homomorphism.

Therefore $\overline{D}fc$ = $I^*(L_f)(\vee\overline{C}(f^{-1}fc \cap C))$ implying that $\vee\overline{C}(f^{-1}fc \cap C) \in I^*(L_f)^{-1}\overline{D}fc$.

Now, since $c \in f^{-1}fc \cap C$, the above implies $\overline{C}c$ ≤ $\vee\overline{C}(f^{-1}fc \cap C)$ ≤ $\vee I^*(L_f)^{-1}$ $\overline{D}fc$ as $\vee\overline{C}(f^{-1}fc \cap C) \in I^*(L_f)^{-1}$ $\overline{D}fc$.

Therefore $\overline{E}c$ = $\overline{A}c \wedge \vee I^*(L_f)^{-1}\overline{D}fc \geq \overline{A}c \wedge \overline{C}c$ = $\overline{C}c$ since $\overline{A}|C \geq \overline{C}$, implying $\overline{E}|C \geq \overline{C}$.

The above proposition is *not* true for decreasing ivf-maps and the Example 4.5.9, serves here also.

Proposition 5.5: *For any 0-p ivf-map $F: A \to B$ and for any $I^*(L_f)$ -regular ivf-subset C of B, we have $F_*F_*^{-1}C \subseteq C$, whenever *=i or d or p.*

Proof: Let $F_*^{-1}C = D$. Then $D = f^{-1}C$, $I^*(L_D) = I^*(L_f)^{-1}I^*(L_C)$ and $\overline{D}d = \overline{A}d \wedge \vee I^*(L_f)^{-1}\overline{C}fd$ for all $d \in D$.

Let F_*D = E . Then E = fD , $I^*(L_E)$ = $(I^*(L_f)I^*(L_D))_{I^*(L_B)}$ and for all $e \in E$, $\overline{E}e$ = $\overline{B}e$ ∧ $\vee I^*(L_f)$ $\overline{D}(f^{-1}e \cap D)$.

We show that $E \subseteq C$ or (a) $E \subseteq C$ (b) $I^*(L_E)$ is a complete ideal of $I^*(L_C)$ and (c) $\overline{E} \leq \overline{C}|E$.

(a): $E = fD = ff^{-1}C \subseteq C$.

(b): $I^*(L_E) = (I^*(L_f)I^*(L_D))_{I^*(L_B)} = (I^*(L_f)I^*(L_f)^{-1} I^*(L_C))_{I^*(L_B)} \subseteq I^*(L_C)_{I^*(L_B)}$ because always $I^*(L_f)I^*(L_f)^{-1}I^*(L_C) \subseteq I^*(L_C)$.

Since $I^*(L_E)$ and $I^*(L_C)$ are complete ideals of $I^*(L_B)$ such that $I^*(L_E) \subseteq I^*(L_C)$, by 3.2.4(c), we get that $I^*(L_E)$ is a complete ideal of $I^*(L_C)$.

(c): Let $e \in E$ be fixed. Then $\overline{E}e = \overline{B}e \wedge \vee I^*(L_f)\overline{D}(f^{-1}e \cap D)$, where $\overline{D}a = \overline{A}a \wedge \vee I^*(L_f)^{-1}\overline{C}fa$.

Now for all $a \in f^{-1}e \cap D$, $fa = e$, $a \in D$ and $I^*(L_f)\overline{D}a$ = $I^*(L_f)\overline{A}a$ ∧ $I^*(L_f)(\vee I^*(L_f)^{-1}\overline{C}fa) \leq I^*(L_f)\overline{A}a \wedge \overline{C}fa \leq \overline{C}e$, for all $a \in f^{-1}e \cap D$, where the first ≤ is by 3.3.11(4) and the fact that F is 0-p. Therefore, $\vee I^*(L_f)\overline{D}(f^{-1}e \cap D) \leq \overline{C}e$ and $\overline{E}e$ = $\overline{B}e \wedge \vee I^*(L_f)\overline{D}(f^{-1}e \cap D) \leq \overline{B}e \wedge \overline{C}e = \overline{C}e$, since $C \subseteq B$ implies $\overline{C} \leq \overline{B}|C$, implying $\overline{E} \leq \overline{C}|E$.

The above proposition is *not* true if F is *not* 0-p and the Example 4.5.11 serves here also.

Proposition 5.6: *For any 0-p ivf-map $F: A \to B$ such that f and $I^*(L_f)$ are one-one and for any ivf-subset C of A, we have C = $F_*^{-1}F_*C$ whenever * = i or p.*

Proof: Let FC = D .Then $D = fC$, $I^*(L_D) = (I^*(L_f)I^*(L_C))_{I^*(L_B)}$ and $\overline{D}d = \overline{B}d \wedge \vee I^*(L_f)\overline{C}(f^{-1}d \cap C)$ for all $d \in D$.

However, since f is one-one, $\overline{D}fc = \overline{B}fc \wedge \vee I^*(L_f)\overline{C}(f^{-1}fc \cap C) = \overline{B}fc \wedge I^*(L_f)\overline{C}c$ for all $c \in C$.

Let $F^{-1}D = E$. Then E = $f^{-1}D$, $I^*(L_E)$ = $I^*(L_f)^{-1}I^*(L_D)$ and $\overline{E}e$ = $\overline{A}e$ ∧ $\vee I^*(L_f)^{-1}\overline{D}fe$ for all $e \in E$.

It is enough to show C = E or (1) $C = E$ (2) $I^*(L_E) = I^*(L_C)$ and (3) $\overline{E} = \overline{C}$.

(a): $E = f^{-1}D = f^{-1}fC = C$, since f is 1-1.

(b): First, by 3.2.3(3), $L_C = [0, \alpha]$ for some $\alpha \in L_A$ and by 4.1.16, $I^*(L_C)$ = $I^*([0,\alpha])$ = $[0, i\alpha]_{I^*(L_A)}$.

By 3.4.3(2) and the above, $I^*(L_D)$ = $(I^*(L_f)I^*(L_C))_{I^*(L_B)}$ = $(I^*(L_f)I^*([0,\alpha]))_{I^*(L_B)}$ =

$(I^*(L_f)[0,i\alpha])_{I^*(L_B)}=[0,I^*(L_f)i\alpha]_{I^*(L_B)}$.

Therefore by 3.4.6(3), since $I^*(L_f)$ is 0-p and $I^*(L_f)i\alpha \in I^*(L_f)I^*(L_A)$,

$I^*(L_E) = I^*(L_f)^{-1}I^*(L_D) = I^*(L_f)^{-1}[0,I^*(L_f)i\alpha]_{I^*(L_B)} = [0,\vee I^*(L_f)^{-1}I^*(L_f)i\alpha]_{I^*(L_B)}$ $= [0,i\alpha] = I^*(L_C)$, where the 4th equality is due to the fact that $I^*(L_f)$ is one-one.

(c): We already have $\overline{C} \le \overline{E}\,|\,C$, because by 6.5.4, $C \subseteq F_*^{-1}F_*C = E$.

Let $e \in E$ be fixed. Then (a) $\overline{D}fe$ above when f is one-one (b) the facts that

(i) $I^*(L_f)\overline{C}e \in I^*(L_f)I^*(L_C) \subseteq I^*(L_f)I^*(L_A)$

(ii) $I^*(L_f)^{-1}$ is join increasing by 3.3.2

(iii) $\overline{B}fe \wedge I^*(L_f)\overline{C}e \le I^*(L_f)\overline{C}e$ (iv) $\overline{C} \le \overline{A}\,|\,C$ and (c) $I^*(L_f)$ is one-one, imply that

$\overline{E}e = \overline{A}e \wedge \vee I^*(L_f)^{-1}\overline{D}fe = \overline{A}e \wedge \vee I^*(L_f)^{-1}(\overline{B}fe \wedge I^*(L_f)\overline{C}e) \le \overline{A}e \wedge \vee I^*(L_f)^{-1}(I^*(L_f)\overline{C}e) = \overline{A}e \wedge \overline{C}e = \overline{C}e$, which in turn implies $\overline{E} \le \overline{C}\,|\,E$.

The above proposition is *not* true if only $I^*(L_f)$ is one-one but f is *not* and the Example 4.5.14 serves here also.

The above proposition is *not* true if the ivf-map is decreasing and both f and $I^*(L_f)$ are bijections and the Example 4.5.15 serves here also.

Proposition 5.7: *For any 0-p ivf-map $F : A \to B$ such that f and $I^*(L_f)$ are onto, and for any ivf-subset D of B, we have $F_*F_*^{-1}D = D$ whenever $*$ = d or p.*

Proof: Let $C = F^{-1}D$. Then $C = f^{-1}D$, $I^*(L_C) = I^*(L_f)^{-1}I^*(L_D)$ and $\overline{C}c = \overline{A}c \wedge \vee I^*(L_f)^{-1}\overline{D}fc$ for all $c \in C$.

Let $E = FC$. Then $E = fC$, $I^*(L_E) = (I^*(L_f)I^*(L_C))_{I^*(L_B)}$ and

$\overline{E}e = \overline{B}e \wedge \vee I^*(L_f)\overline{C}(f^{-1}e \cap C)$

for all $e \in E$.

We will show that $D = E$ or (a) $D = E$ (b) $I^*(L_D) = I^*(L_E)$ and (c) $\overline{D} = \overline{E}$

(a): $E = fC = ff^{-1}D = D$, since f is onto.

(b): $I^*(L_E) = (I^*(L_f)I^*(L_C))_{I^*(L_B)} = (I^*(L_f)I^*(L_f)^{-1}I^*(L_D))_{I^*(L_B)} = (I^*(L_D))_{I^*(L_B)} = I^*(L_D)$, where the third equality is due to $I^*(L_f)$ being onto and the fourth equality is due to $I^*(L_D)$ being a complete ideal of $I^*(L_B)$.

(c): Let $e \in E = C$ be fixed. Since F is decreasing and $D \subseteq B$, we have $\overline{B}f \le I^*(L_f)\overline{A}$ and $\overline{D} \le \overline{B}\,|\,D$. Consequently, for all $c \in f^{-1}e \cap C$, $e = fc$, $c \in C$ and $\overline{D}fc \le \overline{B}fc \le I^*(L_f)\overline{A}c$.

Further, since $I^*(L_f)$ is onto, $\overline{D}fc \in I^*(L_D) \subseteq I^*(L_B) = I^*(L_f)I^*(L_A)$, by 3.3.11(3),

$I^*(L_f)(\vee I^*(L_f)^{-1}\overline{D}fc) = \overline{D}fc$ and hence

$I^*(L_f)\overline{C}c = I^*(L_f)(\overline{A}c \wedge \vee I^*(L_f)^{-1}\overline{D}fc) = I^*(L_f)\overline{A}c \wedge I^*(L_f)(\vee I^*(L_f)^{-1}\overline{D}fc) = I^*(L_f)\overline{A}c \wedge \overline{D}fc = \overline{D}fc = \overline{D}e$, implying $\vee I^*(L_f)\overline{C}(f^{-1}e \cap C) = \overline{D}e$.

Now $\overline{E}e = \overline{B}e \wedge \vee I^*(L_f)\overline{D}(f^{-1}e \cap C) = \overline{B}e \wedge \overline{D}e = \overline{D}e$, because $\overline{D} \le \overline{B}\,|\,D$.

The above proposition is *not* true if F is increasing and both f and $I^*(L_f)$ are bijections and Example 4.5.17 serves here also.

Also, the above proposition is *not* true if only one of f or $I^*(L_f)$ is onto but *not* both and Examples 4.5.18 and 4.5.19 serve here also.

Let us recall from 6.1.3 that for any family of ivf-subsets $(A_i)_{i\in I}$ of A,

(a) $\cup_{i\in I} A_i$ is defined by the ivf-set B, where

(a) $B = \cup_{i\in I} A_i$ (b) $I^*(L_B) = \vee_{i\in I} I^*(L_{A_i})$ and (c) $\overline{B} : B \to I^*(L_B)$ is defined by $\overline{B}b = \vee_{i\in I_b} \overline{A}_i b$, where $I_b = \{i \in I \mid b \in A_i\}$, for all $b \in B$.

and

(b) $\cap_{i\in I} A_i$ is defined by the ivf-set C, where (1) $C = \cap_{i\in I} A_i$ (2) $I^*(L_C) = \wedge_{i\in I} I^*(L_{A_i})$

(c) $\overline{C} : C \to I^*(L_C)$ is defined by $\overline{C}c = \wedge_{i\in I} \overline{A}_i c$ for all $c \in C$.

Proposition 5.8: *For any 0-p ivf-map $F : A \to B$ and for any family of ivf-subsets $(C_j)_{j\in J}$ of A, we have*

$F_*(\cup_{j\in J} C_j) = \cup_{j\in J} F_* C_j$ *whenever* $* = i$ *or* d *or* p *and* L_B *is a complete infinite distributive lattice.*

Proof: Let $C = \cup_{j\in J} C_j$. Then $C = \cup_{j\in J} C_j$, $I^*(L_C) = \vee_{j\in J} I^*(L_{C_j})$ and $\overline{C}c = \vee_{j\in I_c} \overline{C}_j c$, where $I_c = \{j \in J \mid c \in C_j\}$ for all $c \in C$.

Let $D = FC$. Then $D = fC$, $I^*(L_D) = (I^*(L_f) I^*(L_C))_{I^*(L_B)}$ and

$\overline{D}d = \overline{B}d \wedge \vee I^*(L_f)\overline{C}(f^{-1}d \cap C)$ for all $d \in D$.

Let $E_j = FC_j$. Then $E_j = fC_j$, $I^*(L_{E_j}) = (I^*(L_f) I^*(L_{C_j}))_{I^*(L_B)}$ and

$\overline{E}_j e = \overline{B}e \wedge \vee I^*(L_f)\overline{C}_j(f^{-1}e \cap C_j)$ for all $e \in E_j$.

Let $E = \cup_{j\in J} E_j$. Then $E = \cup_{j\in J} E_j$, $I^*(L_E) = \vee_{j\in J} I^*(L_{E_j})$ and $\overline{E}e = \vee_{j\in I_e} \overline{E}_j e$, where $I_e = \{j \in J \mid e \in E_j\}$, for all $e \in E$.

Now we show that $D = E$ or (a) $D = E$ (b) $I^*(L_D) = I^*(L_E)$ and (c) $\overline{D} = \overline{E}$.

(a):

$D = fC = f(\cup_{j\in J} C_j) = \cup_{j\in J} fC_j = \cup_{j\in J} E_j = E$.

(b): First, since F is 0-p, by definition $I^*(L_f)$ is 0-p. But then by 4.2.5(1) L_f is 0-p.

By 4.2.7, $I^*(L_D) = (I^*(L_f) I^*(L_C))_{I^*(L_B)} = I^*((L_f L_C)_{L_B})$ and

$I^*(L_{E_j}) = (I^*(L_f) I^*(L_{C_j}))_{I^*(L_B)} = I^*((L_f L_{C_j})_{L_B})$.

By 4.1.16, $I^*(L_C) = \vee_{j\in J} I^*(L_{C_j}) = I^*(\vee_{j\in J} L_{C_j})$ and

$I^*(L_E) = \vee_{j\in J} I^*(L_{E_j}) = I^*(\vee_{j\in J} L_{E_j})$.

Therefore by 4.1.12, the above imply $L_D = (L_f L_C)_{L_B}$, $L_{E_j} = (L_f L_{C_j})_{L_B}$, $L_C = \vee_{j\in J} L_{C_j}$ and $L_E = \vee_{j\in J} L_{E_j}$.

Again by 4.1.12, to show $I^*(L_D) = I^*(L_E)$, it is enough to show $L_D = L_E$. But $L_D = (L_f L_C)_{L_B} = L_f(\vee_{j\in J} L_{C_j})_{L_B}$, $L_E = \vee_{j\in J} L_{E_j} = \vee_{j\in J} (L_f L_{C_j})_{L_B}$ and as in the f-set-theory setup 4.5.20, $L_D = L_E$, since L_f is 0-p.

(c): Let $y \in fC = f(\cup_{j\in J} C_j)$, $U_x = \{j \in J \mid x \in C_j\}$ and $V_y = \{j \in J \mid y \in fC_j\}$.

Then for all $x \in f^{-1}y \cap C$, $U_x \neq \phi$, $V_y \neq \phi$, $fx = y$ and $x \in C$.

Further, $\overline{D}y = \overline{B}y \wedge \vee I^*(L_f)\overline{C}(f^{-1}y \cap C) = \overline{B}y \wedge \vee_{x\in f^{-1}y\cap C} I^*(L_f)\overline{C}x$

$= \overline{B}y \wedge \vee_{x\in f^{-1}y\cap C} I^*(L_f)\ (\vee_{i\in U_x} \overline{C}_i x) = \overline{B}y \wedge \vee_{x\in f^{-1}y\cap C} \vee_{i\in U_x} I^*(L_f)\overline{C}_i x$.

On the other hand, since L_B is a complete infinite meet distributive lattice,

$\overline{E}y = \vee_{j\in V_y} \overline{E}_j y = \vee_{j\in V_y} (\overline{B}y \wedge \vee_{z\in f^{-1}y\cap C_j} I^*(L_f)\overline{C}_j z) = \overline{B}y \wedge \vee_{j\in V_y} \vee_{z\in f^{-1}y\cap C_j} I^*(L_f)\overline{C}_j z$.

Therefore it is enough to show that

$\vee_{x\in f^{-1}y\cap C} \vee_{i\in U_x} I^*(L_f)\overline{C}_i x = \vee_{j\in V_y} \vee_{z\in f^{-1}y\cap C_j} I^*(L_f)\overline{C}_j z$.

Let $Q = \{I^*(L_f)\overline{C}_j z \mid z \in f^{-1}y \cap C_j, j \in V_y\}$ and $P = \{I^*(L_f)\overline{C}_i x \mid x \in f^{-1}y \cap C, i \in U_x\}$.

Then clearly, it is enough to show that $P = Q$, because $\vee P = \vee_{x\in f^{-1}y\cap C} \vee_{i\in U_x} I^*(L_f)\overline{C}_i x$ and $\vee Q = \vee_{j\in V_y} \vee_{z\in f^{-1}y\cap C_j} I^*(L_f)\overline{C}_j z$.

Let $\alpha \in Q$. Then $\alpha = I^*(L_f)\overline{C}_j z$, $z \in f^{-1}y \cap C_j$, $j \in V_y$. Since $C_j \subseteq C$, $z \in f^{-1}y \cap C$, $j \in U_z$. Therefore $z \in f^{-1}y \cap C$, $j \in U_z$ or $\alpha = I^*(L_f)\overline{C}_j z \in P$, implying $Q \subseteq P$.

Let $\beta \in P$. Then $\beta = I^*(L_f)\overline{C}_i x$, $x \in f^{-1}y \cap C$, $i \in U_x$. But then $x \in f^{-1}y$ and $x \in C_i$ or $x \in f^{-1}y \cap C_i$ which implies $y = fx \in fCi$ or $i \in V_y$ which in turn implies $x \in f^{-1}y \cap C_i$, $i \in V_y$ or $\beta = I^*(L_f)\overline{C}_i x \in Q$, implying $P \subseteq Q$.

Proposition 5.9: *For any 1-p ivf-map* $F: A \to B$ and for any family of ivf-subsets $(C_j)_{j\in J}$ of A, we have $F_*(\cap_{j\in J} C_j) \subseteq \cap_{j\in J} F_* C_j$ whenever $* = i$ or d or p.

Proof: Let $C = \cap_{j\in J} C_j$. Then $C = \cap_{j\in J} C_j$, $I^*(L_C) = \wedge_{j\in J} I^*(L_{C_j})$ and $\overline{C}c = \wedge_{j\in J} \overline{C}_j c$ for all $c \in C$.

Let $D = FC$. Then $D = fC$, $I^*(L_D) = (I^*(L_f)I^*(L_C))_{I^*(L_B)}$ and $\overline{D}d = \overline{B}d \wedge \vee I^*(L_f)\overline{C}(f^{-1}d \cap C)$ for all $d \in D$.

Let $E_j = FC_j$. Then $E_j = fC_j$, $I^*(L_{E_j})$ = $(I^*(L_f)I^*(L_{C_j}))_{I^*(L_B)}$ and $\overline{E}_j e = \overline{B}e \wedge \vee I^*(L_f)\overline{C}_j(f^{-1}e \cap C_j)$ for all $e \in E_J$.

Let $E = \cap_{j\in J} E_j$. Then $E = \cap_{j\in J} E_j$, $I^*(L_E)$ = $\cap_{j\in J} I^*(L_{E_j})$ and $\overline{E}e = \wedge_{j\in J} \overline{E}_j e$ for all $e \in E$.

We will show that $D \subseteq E$ or (a) $D \subseteq E$ (b) $I^*(L_D)$ is a complete deal of $I^*(L_E)$ and (c) $\overline{D} \le \overline{E} \mid D$

(a):

$D = fC = f(\cap_{j\in J} C_j) \subseteq \cap_{j\in J} fC_j = \cap_{j\in J} E_j = E$

(b): First by 4.2.5(2), since $I^*(L_f)$ is 1-p, we get that L_f is 1-p.

By 4.2.7, $I^*(L_D)$ = $(I^*(L_f)I^*(L_C))_{I^*(L_B)}$ = $I^*((L_f L_C)_{L_B})$ and $I^*(L_{E_j})$ = $(I^*(L_f)I^*(L_{C_j}))_{I^*(L_B)}$ = $I^*((L_f L_{C_j})_{L_B})$.

By 4.1.16, since $I^*(L_E)$ = $\wedge_{j\in J} I^*(L_{E_j})$ = $I^*(\wedge_{j\in J} L_{E_j})$ and $I^*(L_C)$ = $\wedge_{j\in J} I^*(L_{C_j})$ = $I^*(\wedge_{j\in J} L_{C_j})$.

By 4.1.12, the above implies L_D = $(L_f L_C)_{L_B}$, L_E = $\wedge_{j\in J} L_{E_j}$, L_{E_j} = $(L_f L_{C_j})_{L_B}$ and L_C = $\wedge_{j\in J} L_{C_j}$.

Now as in the Proof of f-set theory setup 4.5.21(2), L_D = L_E because L_f is 1-p and now 4.1.12 implies $I^*(L_D)$ = $I^*(L_E)$.

(c): Let $y \in D = fC = f(\cap_{j\in J} C_j)$ be fixed. Then $\overline{D}y$ = $\overline{B}y \wedge \vee I^*(L_f)\overline{C}(f^{-1}y \cap C)$ = $\overline{B}y \wedge \vee_{x\in f^{-1}y\cap C} I^*(L_f)\overline{C}x$ and $\overline{E}y$ = $\wedge_{j\in J} \overline{E}_j y$ = $\wedge_{j\in J} (\overline{B}y \wedge \vee I^*(L_f)\overline{C}_j(f^{-1}y \cap C_j))$.

But by 3.1.1(3), $\wedge_{j\in J} (\overline{B}y \wedge \vee I^*(L_f)\overline{C}_j (f^{-1}y \cap C_j))$ = $\overline{B}y$ $\wedge$ $\wedge_{j\in J} \vee I^*(L_f)\overline{C}_j(f^{-1}y \cap C_j)$.

There fore $\overline{E}y$ = $\overline{B}y$ $\wedge$ $\wedge_{j\in J} \vee I^*(L_f)\overline{C}_j(f^{-1}y \cap C_j)$ = $\overline{B}y$ $\wedge$ $\wedge_{j\in J} \vee_{x\in f^{-1}y\cap C_j} I^*(L_f)\overline{C}_j x$.

Also $f^{-1}y \cap C$ = $f^{-1}y \cap (\cap_{j\in J} C_j)$ = $\cap_{j\in J} (f^{-1}y \cap C_j)$ $\subseteq f^{-1}y \cap C_j$ for all $j \in J$, since $A \cap (\cap_{i\in I} B_i) = \cap_{i\in I} (A \cap B_i)$.

Next for all $x \in f^{-1}y \cap C$, $x \in f^{-1}y \cap C_j$ for all $j \in J$ and $\overline{C}x \le \overline{C}_j x$ implying $I^*(L_f)\overline{C}x \le I^*(L_f)\overline{C}_j x \le \vee_{x\in f^{-1}y\cap C} I^*(L_f)\overline{C}_j x$ $\le \vee_{x\in f^{-1}y\cap C_j} I^*(L_f)\overline{C}_j x$ for all $j \in J$ which in turn implies $I^*(L_f)\overline{C}x \le \wedge_{j\in J}(\vee_{x\in f^{-1}y\cap C_j} I^*(L_f)\overline{C}_j x)$ for all $x \in f^{-1}y \cap C$, from which follows:

$\vee_{x\in f^{-1}y\cap C} I^*(L_f)\overline{C}x \le \wedge_{j\in J}(\vee_{x\in f^{-1}y\cap C_j} I^*(L_f)\overline{C}_j x)$.

Therefore, $\overline{D}y$ = $\overline{B}y$ $\wedge$ $\vee_{x\in f^{-1}y\cap C} I^*(L_f)\overline{C}x$ $\le$ $\overline{B}y$ $\wedge$ $\wedge_{j\in J} (\vee_{x\in f^{-1}y\cap C_j} I^*(L_f)\overline{C}_j x)$ = $\overline{E}y$ for all $y \in D$, implying $\overline{D} \le \overline{E} \mid D$ or finally $D \subseteq E$.

Proposition 5.10: *For any 0-p and 0-r ivf-map $F : A \to B$ and for any family of ivf-subsets $(C_j)_{j\in J}$ of B, we have* $F_*^{-1}(\cup_{j\in J} C_j) = \cup_{j\in J} F_*^{-1}C_j$ whenever

(a) $I^*(L_B)$ is a finite chain, L_A is complete infinite meet distributive lattice.

(b) C_j is $I^*(L_f)$-regular for each $j \in J$ and * = i or d or p.

Proof: Let $C = \cup_{j\in J} C_j$. Then $C = \cup_{j\in J} C_j$, $I^*(L_C) = \vee_{j\in J} I^*(L_{C_j})$ and $\overline{C}c = \vee_{j\in I_c} \overline{C}_j c$, where $I_c = \{ j \in J \mid c \in C_j \}$, for all $c \in C$.

Let $D = F_*^{-1}C$. Then $D = f^{-1}C$, $I^*(L_D) = I^*(L_f)^{-1} I^*(L_C)$ and $\overline{D}d$ = $\overline{A}d \wedge \vee I^*(L_f)^{-1}\overline{C}fd$ for all $d \in D$.

Let E_j = $F_*^{-1}C_j$. Then E_j = $f^{-1}C_j$, $I^*(L_{E_j})$ = $I^*(L_f)^{-1} I^*(L_{C_j})$ and $\overline{E}_j e$ = $\overline{A}e \wedge \vee I^*(L_f)^{-1}\overline{C}fe$ for all $e \in E_J$.

Let $E = \cup_{j\in J} E_j$ .Then $E = \cup_{j\in J} E_j$, $I^*(L_E)$ = $\vee_{j\in J} I^*(L_{E_j})$ and $\overline{E}e = \vee_{j\in I_e} \overline{E}_j e$, where $I_e = \{ j \in J \mid e \in E_j \}$, for all $e \in E$.

We will show that $D = E$ or (a) $D = E$ (b) $I^*(L_D) = I^*(L_E)$ and (c) $\overline{D} = \overline{E}$.

(a): D = $f^{-1}C$ = $f^{-1}(\cup_{j\in J} C_j)$ = $\cup_{j\in J} f^{-1}C_j$ = $\cup_{j\in J} E_j = E$.

(b): First by 4.2.5, $I^*(L_f)$ is 0-p and 0-r implies L_f is 0-p or 0-r.

Next, C_j being $I^*(L_f)$ -regular and 4.2.7 imply, $I^*(L_{C_j}) \subseteq I^*(L_f) I^*(L_A)$ = $I^*(L_f L_A)$,

which by 4.1.11 implies that $L_{C_j} \subseteq L_f L_A$.

By 4.2.8, $I^*(L_D) = I^*(L_f)^{-1} I^*(L_C)$ = $I^*(L_f^{-1} L_C)$ and $I^*(L_{E_j}) = I^*(L_f)^{-1} I^*(L_{C_j})$ = $I^*(L_f^{-1} L_{C_j})$.

By 4.1.12 and 4.1.16, the above implies L_D = $L_f^{-1} L_C$, L_{E_j} = $L_f^{-1} L_{C_j}$, L_C = $\vee_{j\in J} L_{C_j}$ and L_E = $\vee_{j\in J} L_{E_j}$.

But then as in the Proof of f-set theory setup 4.5.22(2), since L_f is 0-p, 0-r, L_B is finite chain, L_A is complete infinite meet distributive lattice and $L_{C_j} \subseteq L_f L_A$ for all $j \in J$, L_D = L_E and hence $I^*(L_D)$ = $I^*(L_E)$.

(c): First, by 4.2.5(3), since $I^*(L_f)$ is 0-r, L_f is 0-r.

Next, by 4.1.15(1), since L_A is complete infinite meet distributive lattice, $I^*(L_A)$ is a complete infinite meet distributive lattice.

Now let $x \in D$ = $f^{-1}C$ = $f^{-1}(\cup_{j\in J} C_j)$ be fixed. Then $\overline{D}x$ = $\overline{A}x \wedge \vee I^*(L_f)^{-1}\ \overline{C} fx$ = $\overline{A}x \wedge \vee I^*(L_f)^{-1}\ (\vee_{j\in I_{fx}} \overline{C}_j fx)$ = $\overline{A}x \wedge \vee_{j\in I_{fx}} \vee I^*(L_f)^{-1}\ \overline{C}_j fx$, where the last equality is due to 3.3.19, because of (i) L_B is a finite chain and (ii) L_f is 0-r, where I_{fx} = $\{ j \in J \mid fx \in C_j \}$.

On the other hand, since $I^*(L_A)$ is a complete infinite meet distributive lattice, $\overline{E}x$ = $\vee_{j\in I_x} \overline{E}_j x$ = $\vee_{j\in I_x} (\overline{A}x \wedge \vee I^*(L_f)^{-1} \overline{C}_j fx)$ = $\overline{A}x \wedge \vee_{j\in I_x} \vee I^*(L_f)^{-1} \overline{C}_j fx$, where $I_x = \{ j \in J \mid x \in E_j \}$.

From the above, it is enough to show that $\vee_{j\in I_{fx}} \vee I^*(L_f)^{-1} \overline{C}_j fx$ = $\vee_{k\in I_x} \vee I^*(L_f)^{-1} \overline{C}_k fx$, where $I_{fx} = \{ j \in J \mid fx \in C_j \}$, $I_x = \{ k \in J \mid x \in E_k = f^{-1}C_k \}$.

Clearly it is enough to show that $I_{fx} = I_x$.

Let $j \in I_{fx}$. Then $fx \in C_j$ which implies $x \in f^{-1}C_j = E_j$, implying that $j \in I_x$.

Conversely, $k \in I_x$ implies $x \in E_k = f^{-1}C_k$ which implies $fx \in C_k$ which in turn implies $k \in I_{fx}$.

Therefore I_{fx} = I_x .

The above proposition is *not* true if some C_j is *not* $I^*(L_f)$ -regular but $I^*(L_f)$ is 0-p and 0-r and the Example 4.5.23 serves here also.

Also, the above proposition is *not* true if L_B is *not* a finite chain but F is 0-p and 0-r and the Example 4.5.24. serves here also.

Proposition 5.11: *For any 0-p and 1-p ivf-map* $F : A \to B$ *and for any family of ivf-subsets* $(C_j)_{j\in J}$ *of* B *, we have* $F_*^{-1}(\cap_{j\in J} C_j)$ = $\cap_{j\in J} F_*^{-1} C_j$ whenever C_j is $I^*(L_f)$ -regular for each $j \in J$ and $* = i$ or d or p .

Proof: Let $C = \cap_{j\in J} C_j$. Then $C = \cap_{j\in J} C_j$, $I^*(L_C) = \wedge_{j\in J} I^*(L_{C_j})$ and $\overline{C}c = \wedge_{j\in J} \overline{C}_j c$ for all $c \in C$.

Let D = $F^{-1}C$. Then D = $f^{-1}C$, $I^*(L_D)$ = $I^*(L_f)^{-1} I^*(L_C)$ and $\overline{D}d$ = $\overline{A}d \wedge \vee I^*(L_f)^{-1} \overline{C} fd$ for all $d \in D$.

Let E_j = $F^{-1}C_j$. Then E_j = $f^{-1}C_j$, $I^*(L_{E_j})$ = $I^*(L_f)^{-1} I^*(L_{C_j})$ and $\overline{E}_j e$ = $\overline{A}e \wedge \vee I^*(L_f)^{-1} \overline{C}_j fe$ for all $e \in E_j$.

Let E = $\cap_{j\in J} E_j$ .Then $E = \cap_{j\in J} E_j$, $I^*(L_E)$ = $\wedge_{j\in J} I^*(L_{E_j})$ and $\overline{E}e = \wedge_{j\in J} \overline{E}_j e$ for all $e \in E$.

We show that $D = E$ or (a) $D = E$ (b) $I^*(L_D)$ = $I^*(L_E)$ and (c) $\overline{D} = \overline{E}$.

(a): $D = f^{-1}C = f^{-1}(\cap_{j\in J} C_j) = \cap_{j\in J} f^{-1}C_j = \cap_{j\in J} E_j$ $= E$.

(b): First, by 4.2.5, since F is 0-p, $I^*(L_f)$ and L_f are 0-p and since F is 1-p, $I^*(L_f)$ and L_f are 1-p.

Next, C_j is $I^*(L_f)$ -regular implies $I^*(L_{C_j}) \subseteq I^*(L_f)I^*(L_A)$ = $I^*(L_f L_A)$ which by 4.1.11 implies $L_{C_j} \subseteq L_f L_A$ for all $j \in J$.

By 4.1.16, $I^*(L_C) = \wedge_{j\in J} I^*(L_{C_j})$ = $I^*(\wedge_{j\in J} L_{C_j})$ and $I^*(L_E) = \wedge_{j\in J} I^*(L_{E_j}) = I^*(\wedge_{j\in J} L_{E_j})$.

By 4.2.8, $I^*(L_D) = I^*(L_f)^{-1}I^*(L_C) = I^*(L_f^{-1}L_C)$ and $I^*(L_{E_j}) = I^*(L_f)^{-1}I^*(L_{C_j}) = I^*(L_f^{-1}L_{C_j})$ for all $j \in J$.

By 4.1.12, the above imply $L_C = \wedge_{j\in J} L_{C_j}$, $L_E = \wedge_{j\in J} L_{E_j}$, $L_D = L_f^{-1}L_C$ and $L_{E_j} = L_f^{-1}L_{C_j}$ for all $j \in J$.

But then, since L_f is 0-p and 1-p and $L_{C_j} \subseteq L_f L_A$ for all $j \in J$, as in (2) of f-set theory setup 4.5.25, we get that $L_D = L_E$ and hence $I^*(L_D) = I^*(L_E)$.

(c): Let $x \in D = f^{-1}C = f^{-1}(\cap_{j\in J} C_j)$ be fixed. Then $\overline{D}x = \overline{A}x \wedge \vee I^*(L_f)^{-1}\ \overline{C}fx$ = $\overline{A}x \wedge \vee I^*(L_f)^{-1}(\wedge_{j\in J}\overline{C}_j fx)$ = $\overline{A}x \wedge \wedge_{j\in J} \vee I^*(L_f)^{-1}\overline{C}_j fx$, where the last equality is due to 3.3.16, since

(i) $I^*(L_f)$ is 1-p and hence it is $(\vee,\wedge)$ complete and (ii) $T = \{\overline{C}_j fx \mid j \in J\} \subseteq \cup_{j\in J} I^*(L_{C_j}) \subseteq I^*(L_f)I^*(L_A)$, because each C_j is $I^*(L_f)$ -regular.

On the other hand, by 3.1.1(3), $\overline{E}x = \wedge_{j\in J} \overline{E}_j x$ = $\wedge_{j\in J} (\overline{A}x \wedge \vee I^*(L_f)^{-1}\overline{C}_j fx)$ = $\overline{A}x \wedge \wedge_{j\in J} \vee I^*(L_f)^{-1}\overline{C}_j fx$, implying $\overline{D}x = \overline{E}x$, from the above.

The above proposition is *not* true if some C_j is *not* $I^*(L_f)$ -regular but F is 0-p and 1-p. The Example 4.5.26 serves here also.

Proposition 5.12: *For any pair of ivf-maps* $F: A \to B$ *and* $G: B \to C$ *and for any ivf-subset* E *of* A, *the following are true:*

(a) $(G_*F_i)E = G_*(F_iE)$

(b) $(G_dF_*)E = G_d(F_*E)$, whenever L_C is a complete infinite meet distributive lattice

(c) $(G_pF_p)E = G_p(F_pE)$, whenever L_C is a complete infinite meet distributive lattice.

Proof: Let $(GF)E = H$. Then $H = gfE$, $I^*(L_H) = (I^*(L_gL_f)I^*(L_E))_{I^*(L_C)}$ and $\overline{H}h = \overline{C}h \wedge \vee I^*(L_g)I^*(L_f)\overline{E}((gf)^{-1}h \cap E)$ for all $h \in H$.

Let $FE = I$. Then $I = fE$, $I^*(L_I) = (I^*(L_f)I^*(L_E))_{I^*(L_B)}$ and $\overline{I}i = \overline{B}i \wedge \vee I^*(L_f)\overline{E}(f^{-1}i \cap E)$ for all $i \in I$.

Let $GI = K$. Then $K = gI$, $I^*(L_K) = (I^*(L_g)I^*(L_I))_{I^*(L_C)}$ and $\overline{K}k = \overline{C}k \wedge \vee I^*(L_g)\overline{I}(g^{-1}k \cap I)$ for all $k \in K$.

(a): We show that $H = K$ or (1) $H = K$ (2) $I^*(L_H) = I^*(L_K)$ and (3) $\overline{H} = \overline{K}$.

(a): $H = gfE = g(fE) = gI = K$.

(b): By 4.2.7, $I^*(L_I) = (I^*(L_f)I^*(L_E))_{I^*(L_B)}$ = $I^*((L_fL_E)_{L_B})$ and by 4.1.12, $L_I = (L_fL_E)_{L_B}$.

Now by 3.2.3(3), $L_E = [0,\alpha]$ for some $\alpha \in L_A$. By 3.4.3(2), $L_I = (L_fL_E)_{L_B} = (L_f[0,\alpha])_{L_B} = [0, L_f\alpha]$.

Again by 4.2.7, $I^*(L_K) = (I^*(L_g)I^*(L_I))_{I^*(L_C)}$ = $I^*((L_gL_I)_{L_C})$ and by 4.1.12, $L_K = (L_gL_I)_{L_C}$.

Now by 3.4.3(2), $L_K = (L_gL_I)_{L_C} = (L_g[0, L_f\alpha])_{L_C}$ = $[0, L_gL_f\alpha]$.

On the other hand, by 4.2.7, $I^*(L_H)$ = $(I^*(L_gL_f)I^*(L_E))_{I^*(L_C)} = I^*(((L_gL_f)(L_E))_{L_C})$ and by 4.1.12,

$L_H = (L_gL_f(L_E))_{L_C}$.

Again by 3.4.3(2), $L_H = (L_gL_fL_E)_{L_C}$ = $(L_gL_f[0,\alpha])_{L_C} = [0, L_gL_f\alpha]$.

Clearly, $L_K = L_H$ and hence $I^*(L_K) = I^*(L_H)$.

(c): Let $y \in I = fE$ be fixed. Since F is increasing and $E \subseteq A$, we get that $\overline{B}f \geq I^*(L_f)\overline{A} \geq I^*(L_f)\overline{E}$, and hence for any $x \in f^{-1}y \cap E$, $fx = y$, $x \in E$ and $I^*(L_f)\overline{E}x \leq I^*(L_f)\overline{A}x \leq \overline{B}fx = \overline{B}y$, implying that $\vee I^*(L_f)\overline{E}(f^{-1}y \cap E) \leq \overline{B}y$ or $\overline{I}y = \overline{B}y \wedge \vee I^*(L_f)\overline{E}(f^{-1}y \cap E)$ = $\vee I^*(L_f)\overline{E}(f^{-1}y \cap E)$, for all $y \in I$.

Let $z \in H = gfE$ be fixed. Then $\overline{H}z = \overline{C}z \wedge \vee I^*(L_g)I^*(L_f)\overline{E}((gf)^{-1}z \cap E)$ and $\overline{K}z = \overline{C}z \wedge \vee I^*(L_g)\overline{I}(g^{-1}z \cap I) = \overline{C}z \wedge \vee_{y \in g^{-1}z \cap I} I^*(L_g)\overline{I}y$.

Since (i) $z \in H$ implies $z = gfx$ for some $x \in E$, implying $E \cap (gf)^{-1}z \neq \phi$, $fx \in g^{-1}z \cap I$ implying $g^{-1}z \cap I \neq \phi$ and $x \in f^{-1}y \cap E$ implying $f^{-1}y \cap E \neq \phi$ where $y = fx$
(ii) F is increasing
(iii) $E \subseteq A$
(iv) $(gf)^{-1}z \cap E = \cup_{y \in g^{-1}z \cap fE} f^{-1}y \cap E$
(v) $\vee_{\alpha \in \cup_{i \in I} A_i} \alpha = \vee_{i \in I} \vee_{\alpha \in A_i} \alpha$, we get that

$\overline{K}z = \overline{C}z \wedge \vee_{y \in g^{-1}z \cap I} I^*(L_g)(\vee_{x \in f^{-1}y \cap E} I^*(L_f)\overline{E}x) = \overline{C}z \wedge \vee_{y \in g^{-1}z \cap I} \vee_{x \in f^{-1}y \cap E} I^*(L_g)I^*(L_f)\overline{E}x$
$= \overline{C}z \wedge \vee_{x \in \cup_{y \in g^{-1}z \cap fE} f^{-1}y \cap E} I^*(L_g)I^*(L_f)\overline{E}x = \overline{C}z \wedge \vee_{x \in ((gf)^{-1}z \cap E)} I^*(L_g)I^*(L_f)\overline{E}x$
$= \overline{C}z \wedge \vee I^*(L_g)I^*(L_f)\overline{E}((gf)^{-1}z \cap E) = \overline{H}z$.

(b): Let H, I, K be as in (a) above. We show that $K = H$ or (1) $K = H$ (2) $I^*(L_K) = I^*(L_H)$ and
(c) $\overline{K} = \overline{H}$. Now (1) and (2) follow as in (a).
(3): Let $z \in H = gfE$ be fixed. Then $\overline{H}z = \overline{C}z \wedge \vee I^*(L_g)I^*(L_f)\overline{E}((gf)^{-1}z \cap E)$ and $\overline{K}z = \overline{C}z \wedge \vee I^*(L_g)\overline{I}(g^{-1}z \cap I) = \overline{C}z \wedge \vee_{y \in g^{-1}z \cap I} I^*(L_g)\overline{I}y$.

Since G is decreasing, $\overline{C}g \leq I^*(L_g)\overline{B}$. So, for each $y \in g^{-1}z \cap I$, $gy = z$, $y \in I$ and $\overline{C}z = \overline{C}gy \leq I^*(L_g)\overline{B}y$, implying $\overline{C}z \wedge I^*(L_g)\overline{B}y = \overline{C}z$.

Let $c = \overline{C}z$, $a_y = I^*(L_g)\overline{B}y$, $b_y = \vee_{x \in f^{-1}y \cap E} I^*(L_g)I^*(L_f)\overline{E}x$ and $Y = g^{-1}z \cap I$. Then $c \wedge a_y = c$ for $y \in g^{-1}z$ where $c = \overline{C}z$.

Again since (i) $z \in H$ implies $z = gfx$ for some $x \in E$, implying $E \cap (gf)^{-1}z \neq \phi$, $fx \in g^{-1}z \cap I$ implying $g^{-1}z \cap I \neq \phi$ and $x \in f^{-1}y \cap E$ implying $f^{-1}y \cap E \neq \phi$ where $y = fx$
(ii) $I^*(L_C)$ is a complete infinite meet distributive lattice
(iii) $(gf)^{-1}z \cap E = \cup_{y \in g^{-1}z \cap fE} f^{-1}y \cap E$
(iv) $\vee_{\alpha \in \cup_{i \in I} A_i} \alpha = \vee_{i \in I} \vee_{\alpha \in A_i} \alpha$
from the above we get that $\overline{K}z = \overline{C}z \wedge \vee_{y \in g^{-1}z \cap I} I^*(L_g)(\overline{B}y \wedge \vee_{x \in f^{-1}y \cap E} I^*(L_f)\overline{E}x)$
$= \overline{C}z \wedge \vee_{y \in g^{-1}z \cap I} (I^*(L_g)\overline{B}y \wedge I^*(L_g)(\vee_{x \in f^{-1}y \cap E} I^*(L_f)\overline{E}x))$
$= \overline{C}z \wedge \vee_{y \in g^{-1}z \cap I} (I^*(L_g)\overline{B}y \wedge \vee_{x \in f^{-1}y \cap E} I^*(L_g)I^*(L_f)\overline{E}x)$
$= c \wedge \vee_{y \in Y}(a_y \wedge b_y) = \vee_{y \in Y}(c \wedge a_y \wedge b_y) = \vee_{y \in Y}(c \wedge b_y) = c \wedge \vee_{y \in Y} b_y$
$= \overline{C}z \wedge \vee_{y \in g^{-1}z \cap I} \vee_{x \in f^{-1}y \cap E} I^*(L_g)I^*(L_f)\overline{E}x$
$= \overline{C}z \wedge \vee_{x \in \cup_{y \in g^{-1}z \cap I} f^{-1}y \cap E} I^*(L_g)I^*(L_f)\overline{E}x = \overline{C}z \wedge \vee_{x \in (gf)^{-1}z \cap E} I^*(L_g)I^*(L_f)\overline{E}x$
$= \overline{C}z \wedge \vee I^*(L_g)I^*(L_f)\overline{E}((gf)^{-1}z \cap E) = \overline{H}z$, implying $\overline{K}z = \overline{H}z$.
(c): The proof follows from that (a) and (b).

Proposition 5.13: *For any pair of ivf-maps $F : A \to B$ and $G : B \to C$ and for any ivf-subset E of C, the following are true:*
(a) $(G_d F_*)^{-1}E \supseteq F_*^{-1}G_d^{-1}E$, whenever E is $I^*(L_g)$-regular.
(b) $(G_* F_i)^{-1}E \subseteq F_i^{-1}(G_*^{-1}E)$, whenever $G_*^{-1}E$ is $I^*(L_f)$-regular and $I^*(L_f)$ is 0-p.
(c) $(G_p F_p)^{-1}E = F_p^{-1}(G_p^{-1}E)$, whenever E is $I^*(L_g)$-regular and $G^{-1}E$ is $I^*(L_f)$-regular and $I^*(L_f)$ is 0-p.

Proof: Let $(G_d F_*)^{-1}E = H$. Then $H = (gf)^{-1}E = f^{-1}g^{-1}E$, $I^*(L_H) = I^*(L_g L_f)^{-1}I^*(L_E)$ and $\overline{H}h = \overline{A}h \wedge \vee(I^*(L_g)I^*(L_f))^{-1}\overline{E}(gf)h$ for all $h \in H$.
Let $G^{-1}E = I$. Then $I = g^{-1}E$, $I^*(L_I) = I^*(L_g)^{-1}I^*(L_E)$ and $\overline{I}i = \overline{B}i \wedge \vee I^*(L_g)^{-1}\overline{E}gi$ for all $i \in I$
Let $F^{-1}I = K$. Then $K = f^{-1}I$, $I^*(L_K) = I^*(L_f)^{-1}I^*(L_I)$ and $\overline{K}k = \overline{A}k \wedge \vee I^*(L_f)^{-1}\overline{I}fk$ for all $k \in K$ We show that $H \supseteq K$ or (1) $H \supseteq K$ (2)

$I^*(L_K)$ is a complete ideal of $I^*(L_H)$ and (3) $\overline{H} \mid K \geq \overline{K}$.

(a): $K = f^{-1}I = f^{-1}g^{-1}E = H$.

(b): By 4.2.8, $I^*(L_K) = I^*(L_f)^{-1}I^*(L_I) = I^*(L_f^{-1}L_I)$, $I^*(L_I) = I^*(L_g)^{-1}I^*(L_E) = I^*(L_g^{-1}L_E)$ and $I^*(L_H) = I^*(L_gL_f)^{-1}I^*(L_E) = I^*((L_gL_f)^{-1}L_E)$.

By 4.1.12, the above implies, $L_K = L_f^{-1}L_I$, $L_I = L_g^{-1}L_E$ and $L_H = (L_gL_f)^{-1}L_E$.

Now clearly from the above $L_H = (L_gL_f)^{-1}L_E = L_f^{-1}L_g^{-1}L_E = L_f^{-1}L_I = L_K$ and hence $I^*(L_H) = I^*(L_K)$.

(c): Let $z \in f^{-1}g^{-1}E$ be fixed. Then $fz \in g^{-1}E = I$, $gfz \in E$,

$\overline{H}z = \overline{A}z \wedge \vee(I^*(L_g)I^*(L_f))^{-1}\overline{E}gfz = \overline{A}z \wedge \vee I^*(L_f)^{-1}I^*(L_g)^{-1}\overline{E}gfz$ and $\overline{K}z = \overline{A}z \wedge \vee I^*(L_f)^{-1}\overline{I}fz = \overline{A}z \wedge \vee I^*(L_f)^{-1}(\overline{B}fz \wedge \vee I^*(L_g)^{-1}\overline{E}gfz)$.

Firstly, E is $I^*(L_g)$-regular implies $I^*(L_E) \subseteq I^*(L_g)I^*(L_B)$, $\overline{E}gfa \in I^*(L_E) \subseteq I^*(L_g)I^*(L_B)$ implies $\overline{E}gfa \in I^*(L_g)I^*(L_B)$.

So, by 3.3.11(3), $I^*(L_g)(\vee I^*(L_g)^{-1}\overline{E}gfa) = \overline{E}gfa$.

Since G is decreasing and $E \subseteq C$, $\overline{E}gfa \leq \overline{C}gfa \leq I^*(L_g)\overline{B}fa$,

$I^*(L_g)\overline{I}fa = I^*(L_g)\overline{B}fa \wedge I^*(L_g)(\vee I^*(L_g)^{-1}\overline{E}gfa) = I^*(L_g)\overline{B}fa \wedge \overline{E}gfa = \overline{E}gfa$, implying $\overline{I}fa \in I^*(L_g)^{-1}\overline{E}gfa$ which implies $I^*(L_f)^{-1}\overline{I}fa \subseteq I^*(L_f)^{-1}I^*(L_g)^{-1}\overline{E}gfa$ which in turn implies $\vee I^*(L_f)^{-1}\overline{I}fa \leq \vee I^*(L_f)^{-1}I^*(L_g)^{-1}\overline{E}gfa$ or $\overline{K}a = \overline{A}a \wedge \vee I^*(L_f)^{-1}\overline{I}fa \leq \overline{A}a \wedge \vee I^*(L_f)^{-1}I^*(L_g)^{-1}\overline{E}gfa = \overline{H}a$.

(b): Let H, I and K be as in (a) above. Then it is enough to show, when F is increasing and 0-p and when $G^{-1}E$ is L_f-regular, that $H \subseteq K$ or (1) $H \subseteq K$ (2) $I^*(L_H)$ is a complete ideal of $I^*(L_K)$ and

(c) $\overline{H} \leq \overline{K} \mid H$.

(a): $H = K$ as in (a) above.

(b): $I^*(L_H) = I^*(L_K)$ again as in (a) above.

(c):Let $a \in H = K = f^{-1}g^{-1}E$ be fixed. Then $gfa \in E$, $fa \in g^{-1}E = I$,

$\overline{H}a = \overline{A}a \wedge \vee I^*(L_f)^{-1}I^*(L_g)^{-1}\overline{E}gfa$ and $\overline{K}a = \overline{A}a \wedge \vee I^*(L_f)^{-1}\overline{I}fa = \overline{A}a \wedge \vee I^*(L_f)^{-1}(\overline{B}fa \wedge \vee I^*(L_g)^{-1}\overline{E}gfa)$.

$gfa \in E$ implies $\overline{E}gfa \in \overline{E}E \subseteq I^*(L_E)$ which implies $I^*(L_g)^{-1}\overline{E}gfa \subseteq I^*(L_g)^{-1}I^*(L_E) = I^*(L_I) \subseteq I^*(L_f)I^*(L_A)$, since $G^{-1}E = I$ is $I^*(L_f)$-regular.

Since $I^*(L_f)$ is 0-p and $D = I^*(L_g)^{-1}\overline{E}gfa \subseteq I^*(L_f)I^*(L_A)$, by 3.3.9,

$I^*(L_f)(\vee I^*(L_f)^{-1} I^*(L_g)^{-1} \overline{E}gfa) = \vee I^*(L_g)^{-1}\overline{E}gfa$ and

$I^*(L_f)\overline{H}a = I^*(L_f)\overline{A}a \wedge I^*(L_f)(\vee I^*(L_f)^{-1}I^*(L_g)^{-1}\overline{E}gfa)$

$= I^*(L_f)\overline{A}a \wedge \vee I^*(L_g)^{-1}\overline{E}gfa \leq \overline{B}fa \wedge \vee I^*(L_g)^{-1} \overline{E}gfa = \overline{I}fa$, where the last inequality is due to the fact that F is increasing and hence $I^*(L_f)\overline{A} \leq \overline{B}f$.

Again $gfa \in E$ implies $fa \in g^{-1}E = I$ which implies $\overline{I}fa \in \overline{I}I \subseteq I^*(L_I) \subseteq I^*(L_f)I^*(L_A)$, since $G^{-1}E = I$ is $I^*(L_f)$-regular.

Since $\overline{I}fa \in I^*(L_f)I^*(L_A)$ and $I^*(L_f)\overline{H}a \leq \overline{I}fa$, as above by 3.3.2, we get that $\vee I^*(L_f)^{-1}I^*(L_f)\overline{H}a \leq \vee I^*(L_f)^{-1}\overline{I}fa$.

But then $\overline{H}a \in I^*(L_f)^{-1}I^*(L_f)\overline{H}a$ implies $\overline{H}a \leq \vee I^*(L_f)^{-1}I^*(L_f)\overline{H}a \leq \vee I^*(L_f)^{-1}\overline{I}fa$.

Since always $\overline{H}a \leq \overline{A}a$, it follows that $\overline{H}a \leq \overline{K}a$.

(c): Clearly, the proof follows from (a) and (b).

A strict containment in (a) is possible and the Example 4.5.29 serves here also.

The condition that $G^{-1}E$ is $I^*(L_f)$-regular is *not* superfluous in (b) and the Example 4.5.30 serves here also.

The condition that E is $I^*(L_g)$-regular is *not* superfluous in (c) and the Example 4.5.31 also serves here also.

F. More on M-Interval Valued Fuzzy Images and L-Interval Valued Fuzzy Inverse Images:

In this section some more standard properties of the M-ivf-images of L-ivf-subsets under an ivf-map and the L-

ivf-inverse images of M -ivf-subsets under an ivf-map are studied in detail.

Lemma 6.1 *Forany0-p ivf-map* $F: A \to B$ *and for any* $I^*(L_f)$ *-regular ivf-subset* H *of* B *, always* $F^{-1}H \supseteq F^{-1}(H \cap FA)$. *However, equality holds whenever*

(a) F is increasing, $I^*(L_f)$ is 1-p and $I^*(L_B)$ is complete infinite meet distributive lattice (OR)

(b) F is decreasing and $I^*(L_B)$ is complete infinite meet distributive lattice.

Proof: (A) Since H is $I^*(L_f)$ -regular and $H \cap FA \subseteq H$, by 5.5.3, F^{-1} is monotonic and so, $F^{-1}(H \cap FA) \subseteq F^{-1}(H)$.

(B) Let $F^{-1}H = C$. Then $C = f^{-1}H$, $I^*(L_C) = I^*(L_f)^{-1}I^*(L_H)$ and $\overline{C}a = \overline{A}a \wedge \vee I^*(L_f)^{-1}\overline{H}\,fa$ for all $a \in C$.

Let $FA = D$. Then $D = fA$, $I^*(L_D) = (I^*(L_f)I^*(L_A))_{I^*(L_B)}$ and $\overline{D}b = \overline{B}b \wedge \vee I^*(L_f)\overline{A}(f^{-1}b \cap A)$ for all $b \in D$.

Let $H \cap D = E$. Then $E = H \cap D$, $I^*(L_E) = I^*(L_H) \cap I^*(L_D)$ and $\overline{E}b = \overline{H}b \wedge \overline{D}b$ for all $b \in E$.

Let $F^{-1}E = G$. Then $G = f^{-1}E$, $I^*(L_G) = I^*(L_f)^{-1}I^*(L_E)$ and $\overline{G}a = \overline{A}a \wedge \vee I^*(L_f)^{-1}\overline{E}\,fa$ for all $a \in G$.

We show that $C = G$ or (1) $C = G$ (2) $I^*(L_C) = I^*(L_G)$ (3) $\overline{C} = \overline{G}$ when

(a) F isincreasing, $I^*(L_f)$ is 1-p and $I^*(L_B)$ is complete infinite meet distributive lattice (OR)

(b) F is decreasing and $I^*(L_B)$ is complete infinite meet distributive lattice.

(a): $C = f^{-1}H = f^{-1}(H \cap fA) = f^{-1}(H \cap D) = f^{-1}E = G$.

(b): First, (i) H is $I^*(L_f)$ -regular implies $I^*(L_H) \subseteq I^*(L_f)I^*(L_A) = I^*(L_f L_A)$,

where the last equality is due to 4.2.7. By 4.1.11, the preceding statement implies $L_H \subseteq L_f L_A$ and

(ii) F is 0-p implies by definition, $I^*(L_f)$ is 0-p which by 4.2.5, implies that L_f is 0-p.

Next, by 4.2.8, and 4.1.12, $I^*(L_C) = I^*(L_f)^{-1}I^*(L_H) = I^*(L_f^{-1}L_H)$ and so $L_C = L_f^{-1}L_H$ and $I^*(L_G) = I^*(L_f)^{-1}I^*(L_E) = I^*(L_f^{-1}L_E)$ and so $L_G = L_f^{-1}L_E$.

By 4.2.7 and 4.1.12, $I^*(L_D) = (I^*(L_f)I^*(L_A))_{I^*(L_B)} = I^*((L_f L_A)_{L_B})$ and so $L_D = (L_f L_A)_{L_B}$ and by 4.1.16 and 4.1.12, $I^*(L_E) = I^*(L_H) \wedge I^*(L_D) = I^*(L_H \wedge L_D)$ and so $L_E = L_H \wedge L_D$.

Now as in 5.6.1(B)(2) above, the above implies that L_G = L_C and hence $I^*(L_G) = I^*(L_C)$.

(c): Let $a \in G = f^{-1}E = C = f^{-1}H$ be fixed. Then $fa \in H \cap E$.

(a): Let F be decreasing.Then $\overline{B}f \le I^*(L_f)\overline{A}$.

Further, for all $c \in f^{-1}fa \cap A$, $I^*(L_f)\overline{A}c \ge \overline{B}fc = \overline{B}fa$ or $\vee I^*(L_f)\overline{A}(f^{-1}fa \cap A) \ge \wedge I^*(L_f)\overline{A}(f^{-1}fa \cap A) \ge \overline{B}fa$, implying $\overline{D}fa = \overline{B}fa \wedge \vee I^*(L_f)\overline{A}(f^{-1}fa \cap A) = \overline{B}fa$ which in turn implies

$\overline{G}a = \overline{A}a \wedge \vee I^*(L_f)^{-1}\overline{E}\,fa = \overline{A}a \wedge \vee I^*(L_f)^{-1}(\overline{H}\,fa \wedge \overline{D}\,fa) = \overline{A}a \wedge \vee I^*(L_f)^{-1}(\overline{H}\,fa \wedge \overline{B}\,fa)$

$= \overline{A}a \wedge \vee I^*(L_f)^{-1}\overline{H}\,fa = \overline{C}a$, because $\overline{E} = \overline{H} \cap \overline{D}$ and $\overline{H} \le \overline{B}$.

(b): Let F be increasing.Then $\overline{B}f \ge I^*(L_f)\overline{A}$.

For all $c \in f^{-1}fa \cap A$, $I^*(L_f)\overline{A}c \le \overline{B}fc = \overline{B}fa$ or $\vee I^*(L_f)\overline{A}(f^{-1}fa \cap A) \le \overline{B}fa$ implying

$\overline{D}fa = \overline{B}fa \wedge \vee I^*(L_f)\overline{A}(f^{-1}fa \cap A) = \vee I^*(L_f)\overline{A}(f^{-1}fa \cap A)$.

Therefore $\overline{E}fa = \overline{H}\,fa \wedge \overline{D}fa = \overline{H}\,fa \wedge \vee I^*(L_f)\overline{A}(f^{-1}fa \cap A)$.

Next, since (i) H is $I^*(L_f)$ -regular and hence $\overline{H}\,fa \in I^*(L_H) \subseteq I^*(L_f)I^*(L_A)$

(ii) $\vee I^*(L_f)\overline{A}(f^{-1}fa \cap A) \in I^*(L_f)I^*(L_A)$ as $f^{-1}fa \cap A \ne \phi$ and (iii) $I^*(L_f)$ is 1-p, by 3.3.15,

$\vee I^*(L_f)^{-1}(\overline{H}\,fa \wedge \vee I^*(L_f)\overline{A}(f^{-1}fa \cap A)) = \vee I^*(L_f)^{-1}\overline{H}\,fa \wedge \vee I^*(L_f)^{-1}(\vee I^*(L_f)\overline{A}(f^{-1}fa \cap A))$ .Further, since

$\vee I^*(L_f)\overline{A}(f^{-1}fa \cap A) \in I^*(L_f)L_A$ as $f^{-1}fa \cap A \neq \phi$ and $\vee I^*(L_f)\overline{A}(f^{-1}fa \cap A) \geq I^*(L_f)\overline{A}a$, by 3.3.2, $\vee I^*(L_f)^{-1}\ (\vee I^*(L_f)\overline{A}\ (f^{-1}fa \cap A))$ $\geq \vee I^*(L_f)^{-1}(I^*(L_f)\overline{A}a) \geq \overline{A}a$, where the last inequality is due to the fact that $\overline{A}a \in I^*(L_f)^{-1}(I^*(L_f)\overline{A}a)$ Consequent from the above, $\overline{G}a = \overline{A}a \wedge \vee I^*(L_f)^{-1}\overline{E}fa = \overline{A}a \wedge \vee I^*(L_f)^{-1}(\overline{H}fa \wedge \vee I^*(L_f)\overline{A}\ (f^{-1}fa \cap A))$ $= \overline{A}a \wedge (\vee I^*(L_f)^{-1}\overline{H}fa \wedge \vee I^*(L_f)^{-1}(\vee I^*(L_f)\overline{A}(f^{-1}fa \cap A)))$ $= (\overline{A}a \wedge \vee I^*(L_f)^{-1}(\vee I^*(L_f)\overline{A}(f^{-1}fa \cap A))) \wedge \vee I^*(L_f)^{-1}\overline{H}fa$ $= \overline{A}a \wedge \vee I^*(L_f)^{-1}\overline{H}fa = \overline{C}a$.

The above Proposition is *not* true if F is decreasing, $I^*(L_B)$ is a complete infinite meet distributive lattice but H is *not* $I^*(L_f)$-regular and the Example 4.6.2 serves here also.

The above Proposition is *not* true if F is increasing, $I^*(L_f)$ is 1-p and $I^*(L_B)$ is a complete infinite meet distributive lattice but H is *not* $I^*(L_f)$-regular and the Example 4.6.3 serves here also.

Lemma 6.2: *For any 0-p ivf-map $F: A \to B$ and for any $I^*(L_f)$-regular ivf-subset Y of B, we have $F_*^{-1}F_*F_*^{-1}Y = F_*^{-1}Y$, whenever $* = i$ or d or p.*

Proof: Let $F^{-1}Y = C$. Then $C = f^{-1}Y$, $I^*(L_C) = I^*(L_f)^{-1}I^*(L_Y)$ and $\overline{C}a = \overline{A}a \wedge \vee I^*(L_f)^{-1}\overline{Y}fa$ for all $a \in C$.

Let $FC = D$. Then $D = fC$, $I^*(L_D) = (I^*(L_f)I^*(L_C))_{I^*(L_B)}$ and $\overline{D}b = \overline{B}b \wedge \vee I^*(L_f)\overline{C}(f^{-1}b \cap C)$ for all $b \in D$.

Let $F^{-1}D = E$. Then $E = f^{-1}D$, $I^*(L_E) = I^*(L_f)^{-1}I^*(L_D)$ and $\overline{E}a = \overline{A}a \wedge \vee I^*(L_f)^{-1}\overline{D}fa$ for all $a \in E$.

We show that $E = C$ or (1) $E = C$ (2) $I^*(L_E) = I^*(L_C)$ and (3) $\overline{E} = \overline{C}$.

(a): $E = f^{-1}D = f^{-1}fC = f^{-1}ff^{-1}B = f^{-1}B = C$, since $f^{-1}ff^{-1}B = f^{-1}B$.

(b): First, since F is 0-p, by definition, $I^*(L_f)$ is 0-p and by 4.2.5, L_f is 0-p.

Next, since Y is $I^*(L_f)$-regular by 4.2.7, $I^*(L_Y) \subseteq I^*(L_f)I^*(L_A) = I^*(L_fL_A)$ and by 4.1.11, $L_Y \subseteq L_fL_A$. By 4.2.8, $I^*(L_C) = I^*(L_f)^{-1}I^*(L_Y) = I^*(L_f^{-1}L_Y)$ and $I^*(L_E) = I^*(L_f)^{-1}I^*(L_D) = I^*(L_f^{-1}L_D)$ and by 4.1.12, the previous statements imply $L_C = L_f^{-1}L_Y$ and $L_E = L_f^{-1}L_D$.

Now by 4.2.7, $I^*(L_D) = (I^*(L_f)I^*(L_C))_{I^*(L_B)} = I^*((L_fL_C)_{L_B})$ and by 4.1.12, $L_D = (L_fL_C)_{L_B}$.

Now as in (2): of 5.6.4, $L_E = L_C$ and hence $I^*(L_E) = I^*(L_C)$.

(3): Let $a \in E = f^{-1}D = C = f^{-1}Y$ be fixed. Then $fa \in Y \cap D$.

(a): Let F be increasing. Since $C \subseteq F_*^{-1}F_*C = E$ for all $C \subseteq A$ when $* = i$ or p, we have $\overline{C} \leq \overline{E}$. Therefore it is enough to show that $\overline{E} \leq \overline{C}$.

But since $\overline{E}a = \overline{A}a \wedge \vee I^*(L_f)^{-1}\overline{D}fa$ and $\overline{C}a = \overline{A}a \wedge \vee I^*(L_f)^{-1}\overline{Y}fa$, it is enough to show that $\vee I^*(L_f)^{-1}\overline{D}fa \leq \vee I^*(L_f)^{-1}\overline{Y}fa$.

Let $c \in f^{-1}fa \cap C$. Then $c \in C$ and $fc = fa$. Further, since Y is $I^*(L_f)$-regular, $\overline{Y}fc = \overline{Y}fa \in I^*(L_Y) \subseteq I^*(L_f)I^*(L_A)$ and hence by 3.3.11(3), $I^*(L_f)\ (\vee I^*(L_f)^{-1}\ \overline{Y}fc) = \overline{Y}fc = \overline{Y}fa$.

Now $I^*(L_f)\overline{C}c = I^*(L_f)(\overline{A}c \wedge \vee I^*(L_f)^{-1}\overline{Y}fc) = I^*(L_f)\overline{A}c \wedge I^*(L_f)(\vee I^*(L_f)^{-1}\overline{Y}fc)$ $= I^*(L_f)\overline{A}c \wedge \overline{Y}fc \leq \overline{Y}fc = \overline{Y}fa$, implying $\vee I^*(L_f)\overline{C}(f^{-1}fa \cap C) \leq \overline{Y}fa$.

Therefore $\overline{D}fa = \overline{B}fa \wedge \vee I^*(L_f)\overline{C}(f^{-1}fa \cap C) \leq \overline{B}fa \wedge \overline{Y}fa = \overline{Y}fa$, because $Y \subseteq B$.

Now, again Y is $I^*(L_f)$-regular and hence $\overline{Y}fa \in I^*(L_f)I^*(L_A)$ and $\overline{D}fa \leq \overline{Y}fa$ imply, by 3.3.2, $\vee I^*(L_f)^{-1}\overline{D}fa \leq \vee I^*(L_f)^{-1}\overline{Y}fa$, as required.

(b): Let F be decreasing. Then $\overline{B}f \leq I^*(L_f)\overline{A}$. Since $Y \subseteq B$, $\overline{Y}f \leq \overline{B}f \leq I^*(L_f)\overline{A}$. Therefore for any $c \in C$, $I^*(L_f)\overline{C}c =$ $I^*(L_f)\overline{A}c \wedge I^*(L_f)(\vee I^*(L_f)^{-1}\overline{Y}fc)$ $= I^*(L_f)\overline{A}c \wedge \overline{Y}fc = \overline{Y}fc = \overline{Y}fa$, because

(i) Y is $I^*(L_f)$ -regular and hence $\overline{Y}fc \in I^*(L_Y) \subseteq I^*(L_f)I^*(L_A)$ and (ii) by3.3.11(3), $I^*(L_f)(\vee I^*(L_f)^{-1}\overline{Y}fc) = \overline{Y}fc$.

In particular, $\vee I^*(L_f)\overline{C}(f^{-1}fa \cap C)$ = $\vee_{c\in f^{-1}fa\cap C} I^*(L_f)\overline{C}c$ = $\vee_{c\in f^{-1}fa\cap C}\overline{Y}fa$ = $\overline{Y}fa$, implying $\overline{D}fa = \overline{B}fa \wedge \vee I^*(L_f)\overline{C}(f^{-1}fa \cap C) = \overline{B}fa \wedge \overline{Y}fa = \overline{Y}fa$, because $Y \subseteq B$ and hence $\overline{Y} \le \overline{B}\,|\,Y$. Now clearly, $\overline{E}a = \overline{A}a \wedge \vee I^*(L_f)^{-1}\overline{D}fa = \overline{A}a \wedge \vee I^*(L_f)^{-1}\overline{Y}fa = \overline{C}a$.

The above Proposition is *not* true if Y is *not* $I^*(L_f)$ -regular and the Example 4.6.5 serves here too.

Definition 6.3: *For any $F : A \to B$ and for any ivf-subset C of A, C is said to be $I^*(L_f)$ -coregular iff* $\overline{B}fC \subseteq I^*(L_f)I^*(L_A)$.

Proposition 6.4: *For any 0-p ivf-map $F : A \to B$ and for any $I^*(L_f)$ -coregular ivf-subset C of A, we have $F_*F_*^{-1}F_*C = F_*C$ holds whenever $*$ = i or d or p.*

Proof: Let $FC = D$. Then $D = fC$, $I^*(L_D) = (I^*(L_f)I^*(L_C))_{I^*(L_B)}$ and

$\overline{D}b = \overline{B}b \wedge \vee I^*(L_f)\ \overline{C}(f^{-1}b \cap C)$ for all $b \in D$.

Let $F^{-1}D = E$. Then $E = f^{-1}D$, $I^*(L_E) = I^*(L_f)^{-1}\ I^*(L_D)$ and $\overline{E}a = \overline{A}a \wedge \vee I^*(L_f)^{-1}\overline{D}fa$ for all $a \in E$.

Let $FE = G$. Then $G = fE$, $I^*(L_G) = (I^*(L_f)I^*(L_E))_{I^*(L_B)}$ and $\overline{G}b = \overline{B}b \wedge \vee I^*(L_f)\overline{E}(f^{-1}b \cap E)$ for all $b \in G$.

we show that $D = G$ or (1) $D = G$ (2) $I^*(L_D) = I^*(L_G)$ and (3) $\overline{D} = \overline{G}$.

(a): $G = fE = ff^{-1}D = ff^{-1}fC = fC = D$.

(b): First, since F is 0-p, by definition, $I^*(L_f)$ is 0-p and by 4.2.5, L_f is 0-p.

By 4.2.7, $I^*(L_D) = (I^*(L_f)I^*(L_C))_{I^*(L_B)} = I^*((L_fL_C)_{L_B})$ and $I^*(L_G) = (I^*(L_f)\ I^*(L_E))_{I^*(L_B)} = I^*((L_fL_E)_{L_B})$ and by 4.1.12, $L_D = (L_fL_C)_{L_B}$ and $L_G = (L_fL_E)_{L_B}$.

By 4.2.8, $I^*(L_E) = I^*(L_f)^{-1}I^*(L_D) = I^*(L_f^{-1}L_D)$ and by 4.1.12, $L_E = L_f^{-1}L_D$.

Now as in (2): of 5.6.7, $L_G = L_D$ and hence $I^*(L_G) = I^*(L_D)$.

(c): Let $b \in G$ (= fE = fC = D) be fixed. Then $f^{-1}b \cap C \ne \phi$ and $f^{-1}b \cap E \ne \phi$.

(a) Let F be decreasing. Then $\overline{B}f \le I^*(L_f)\overline{A}$. Since $D \subseteq B$, $\overline{D} \le \overline{B}\,|\,D$ and hence $\overline{D}f \le \overline{B}f \le I^*(L_f)\overline{A}$ Since(i) $I^*(L_f)\overline{C}(f^{-1}b \cap C) \subseteq I^*(L_f)\overline{C}C \subseteq I^*(L_f)I^*(L_C) \subseteq I^*(L_f)I^*(L_A)$ (ii) $\overline{B}b \in \overline{B}fC \subseteq I^*(L_f)I^*(L_A)$ because C is $I^*(L_f)$ -coregular and (iii) $I^*(L_f)I^*(L_A)$ is a complete sublattice, we get that $\overline{D}b = \overline{B}b \wedge \vee I^*(L_f)\ \overline{C}(f^{-1}b \cap C) \in I^*(L_f)I^*(L_A)$. So, by 3.3.11(3), $I^*(L_f)(\vee I^*(L_f)^{-1}\overline{D}b) = \overline{D}b$.

Now for all $e \in f^{-1}b \cap E$, $fe = b$ and from the above, $I^*(L_f)\overline{E}e = I^*(L_f)\ (\overline{A}e \wedge \vee I^*(L_f)^{-1}\overline{D}fe) = I^*(L_f)\overline{A}e \wedge I^*(L_f)\ (\vee I^*(L_f)^{-1}\overline{D}fe) = I^*(L_f)\overline{A}e \wedge \overline{D}fe = \overline{D}fe = \overline{D}b$, where the last but one equality follows from F being decreasing.

Therefore, $\vee I^*(L_f)\overline{E}(f^{-1}b \cap E) = \vee_{e\in f^{-1}b\cap E} I^*(L_f)\overline{E}e = \vee_{e\in f^{-1}b\cap E}\overline{D}fe = \vee_{e\in f^{-1}b\cap E}\overline{D}b = \overline{D}b$.

On the other hand, $\overline{G}b = \overline{B}b \wedge \vee I^*(L_f)\overline{E}(f^{-1}b \cap E) = \overline{B}b \wedge \overline{D}b = \overline{D}b$,since $D \subseteq B$ and hence $\overline{D} \le \overline{B}\,|\,D$.

(b): Let F be increasing. Then For any increasing ivf-map,by6.5.4, $C \subseteq F_*^{-1}F_*C$ for all $C \subseteq A$. So, by 6.5.2, monotonicity of F_* implies $D = F_*C \subseteq F_*F_*^{-1}F_*C = G$. Hence it is enough to show that $\overline{G} \le \overline{D}$.

For all $e \in f^{-1}b \cap E$, $fe = b$, $fe \in fC(= D = G = fE)$ and as in (a) above, $\overline{D}fe \in I^*(L_f)I^*(L_A)$ and $I^*(L_f)(\vee I^*(L_f)^{-1}\overline{D}fe) = \overline{D}fe = \overline{D}b$.

Now $\overline{E}e \le \vee I^*(L_f)^{-1}\overline{D}fe$ for all $e \in f^{-1}b \cap E$, implying $I^*(L_f)\overline{E}e \le I^*(L_f)(\vee I^*(L_f)^{-1}\overline{D}fe) = \overline{D}fe = \overline{D}b$ and $\overline{G}b = \overline{B}b \wedge \vee I^*(L_f)\overline{E}(f^{-1}b \cap E) \le$

$\vee I^*(L_f)\overline{E}\ \ (f^{-1}b \cap E) = \vee_{e\in f^{-1}b\cap E}\ I^*(L_f)\overline{E}e \le \overline{D}b$ or $\overline{G} \le \overline{D}$.

The above proposition is *not* true if C is *not* $I^*(L_f)$-coregular but F is 0-p and the Example 4.6.8 serves here also.

Proposition 6.5: *For any increasing f-map $F: A \to B$ and for any pair of f-subsets C of A and D of B, $FC \subseteq D$ implies $C \subseteq F^{-1}D$ whenever D is $I^*(L_f)$-regular.*

Proof: Let $FC = E$. Then $E = fC$, $I^*(L_E) = (I^*(L_f)I^*(L_C))_{I^*(L_B)}$ and $\overline{E}b = \overline{B}b \wedge \vee I^*(L_f)\overline{C}(f^{-1}b \cap C)$ for all $b \in E$.

Let $F^{-1}D = G$. Then $G = f^{-1}D$, $I^*(L_G) = I^*(L_f)^{-1}I^*(L_D)$ and $\overline{G}a = \overline{A}a \wedge \vee I^*(L_f)^{-1}\overline{D}fa$ for all $a \in G$.

Since $E \subseteq D$, $E \subseteq D$, $I^*(L_E)$ is a complete ideal of $I^*(L_D)$ and $\overline{E} \le \overline{D}\,|\,E$.

We show that $C \subseteq G$ or (1) $C \subseteq G$ (2) $I^*(L_C)$ is a complete ideal of $I^*(L_G)$ and (3) $\overline{C} \le \overline{G}\,|\,C$.

(a): Since $fC \subseteq D$ iff $C \subseteq f^{-1}D$, $C \subseteq f^{-1}D = G$.

(b): Since $I^*(L_E)$ is a complete ideal of $I^*(L_D)$ and $I^*(L_f)I^*(L_C) \subseteq (I^*(L_f)I^*(L_C))_{I^*(L_B)} = I^*(L_E) \subseteq I^*(L_D)$, we get that $I^*(L_C) \subseteq I^*(L_f)^{-1}I^*(L_D) = I^*(L_G)$.

Since $I^*(L_G)$ and $I^*(L_C)$ are complete ideals of $I^*(L_A)$, it follows from $I^*(L_C) \subseteq I^*(L_G)$ that $I^*(L_C)$ is a complete ideal of $I^*(L_G)$.

(c): Let $a \in C$ be fixed. Then $fa \in fC = E$. $\overline{G}a = \overline{A}a \wedge \vee I^*(L_f)^{-1}\overline{D}fa$. Since $\overline{A}a \ge \overline{C}a$ to show $\overline{C} \le \overline{G}\,|\,C$, it is enough to show that $\vee I^*(L_f)^{-1}\overline{D}fa \ge \overline{C}a$.

Since (i) $a \in f^{-1}fa \cap C$, $I^*(L_f)\overline{C}a \le \vee I^*(L_f)\overline{C}(f^{-1}fa \cap C)$ and (ii) $\overline{E} \le \overline{D}\,|\,E$, we get that $\overline{B}fa \wedge I^*(L_f)\overline{C}a \le \overline{B}fa \wedge \vee I^*(L_f)\overline{C}(f^{-1}fa \cap C) = \overline{E}fa \le \overline{D}fa$.

Since $C \subseteq A$ and F is increasing, $I^*(L_f)\overline{C}a \le I^*(L_f)\overline{A}a \le \overline{B}fa$ which implies $I^*(L_f)\overline{C}a = \overline{B}fa \wedge I^*(L_f)\overline{C}a \le \overline{D}fa$, from the above.

Since (i) $\overline{D}fa \in I^*(L_D) \subseteq I^*(L_f)I^*(L_A)$ as D is $I^*(L_f)$-regular (ii) $I^*(L_f)\overline{C}a \le \overline{D}fa$, by 3.3.2, $\overline{C}a \le \vee I^*(L_f)^{-1}I^*(L_f)\overline{C}a \le \vee I^*(L_f)^{-1}\overline{D}fa$ as required.

The above Proposition is *not* true if D is *not* $I^*(L_f)$-regular but F is increasing and the Example 4.6.10 serves here also.

The above Proposition is *not* true if F is decreasing but D is $I^*(L_f)$-regular and the Example 4.6.11 serves here also.

Proposition 6.6: *For any ivf-map $F: A \to B$ and for any pair of ivf-subsets C of A and D of B, $C \subseteq F^{-1}D$ implies $FC \subseteq D$, whenever F is 0-p or D is $I^*(L_f)$-regular.*

Proof: Let $FC = E$. Then $E = fC$, $I^*(L_E) = (I^*(L_f)I^*(L_C))_{I^*(L_B)}$ and $\overline{E}b = \overline{B}b \wedge \vee I^*(L_f)\overline{C}(f^{-1}b \cap C)$ for all $b \in E$.

Let $F^{-1}D = G$. Then $G = f^{-1}D$, $I^*(L_G) = I^*(L_f)^{-1}I^*(L_D)$ and $\overline{G}a = \overline{A}a \wedge \vee I^*(L_f)^{-1}\overline{D}fa$ for all $a \in G$.

Since $C \subseteq G$, we have $C \subseteq G$, $I^*(L_C)$ is a complete ideal of $I^*(L_G)$ and $\overline{C} \le \overline{G}\,|\,C$.

We show that $E \subseteq D$ or (1) $E \subseteq D$ (2) $I^*(L_E)$ is a complete ideal of $I^*(L_D)$ and (3) $\overline{E} \le \overline{D}\,|\,E$.

(a): $C \subseteq G = f^{-1}D$ implies $fC \subseteq D$ which implies $E \subseteq D$.

(b): Since $I^*(L_C) \subseteq I^*(L_G) = I^*(L_f)^{-1}\ I^*(L_D)$, $I^*(L_f)I^*(L_C) \subseteq I^*(L_D)$ and $I^*(L_D)$ is a complete ideal of $I^*(L_B)$ implies $I^*(L_E) = (I^*(L_f)I^*(L_C))_{I^*(L_B)} \subseteq I^*(L_D)$. Since $I^*(L_E)$ and $I^*(L_D)$ are complete ideals of $I^*(L_B)$ such that $I^*(L_E) \subseteq I^*(L_D)$, we get that $I^*(L_E)$ is a complete ideal of $I^*(L_D)$.

(c): Let $b \in E = fC$ be fixed. For any $a \in f^{-1}b \cap C$, $a \in C$ and $b = fa \in fC = D$.

Since (i) F and hence $I^*(L_f)$ is 0-p, by 3.3.11(4), $I^*(L_f)(\vee I^*(L_f)^{-1}\overline{D}fa) \le \overline{D}fa$ or

(ii) D is $I^*(L_f)$-regular, so $I^*(L_D) \subseteq I^*(L_f)I^*(L_A)$ and hence $\overline{D}fa \in I^*(L_D) \subseteq I^*(L_f)I^*(L_A)$, by 3.3.11(3), $I^*(L_f)(\vee I^*(L_f)^{-1}\overline{D}fa) = \overline{D}fa$.

But as $C \subseteq G$, $\overline{C} \leq \overline{G} | C$ and this implies $I^*(L_f)\overline{C} \leq I^*(L_f)\overline{G}$ and hence from the above, $I^*(L_f)\overline{C}a \leq I^*(L_f)\overline{G}a = I^*(L_f)(\overline{A}a \wedge \vee I^*(L_f)^{-1}\overline{D}fa)$ $= I^*(L_f)\overline{A}a \wedge I^*(L_f)(\vee I^*(L_f)^{-1}\overline{D}fa) \leq I^*(L_f)\overline{A}a \wedge \overline{D}fa \leq \overline{D}fa = \overline{D}b$ for all $a \in f^{-1}b \cap C$, implying $\vee I^*(L_f)\overline{C}(f^{-1}b \cap C) \leq \overline{D}b$ and $\overline{E}b = \overline{B}b \wedge \vee I^*(L_f)\overline{C}(f^{-1}b \cap C) \leq \vee I^*(L_f)\overline{C}(f^{-1}b \cap C) \leq \overline{D}b$, implying $\overline{E} \leq \overline{D}$ or $FC = E \subseteq D$.

The above Proposition is *not* true if both F is *not* 0-p and D is *not* $I^*(L_f)$ -regular and the Example 4.6.13 serves here also.

Lemma 6.7: *For any ivf-map* $F: X \to Y$ *and for any ivf-subset* A *of* X *,* $A = \Phi$ *iff* $FA = \Phi$.

Proof: $(\Rightarrow)$: $A = \Phi$ implies $A = \phi$, $I^*(L_A) = \phi$ and $\overline{A} = \phi$. $FA = C$ implies $C = fA = f\phi = \phi$, $I^*(L_C) = (I^*(L_f)I^*(L_A))_{I^*(L_B)} = \phi$ and $\overline{C} \subseteq C \times I^*(L_C) = \phi$, implying $FA = C = \Phi$.

$(\Leftarrow)$: $FA = C = \Phi$ implies, $C = fA = \phi$ which implies $A = \phi$, since $fA = \phi$ iff $A = \phi$, $I^*(L_f)I^*(L_A) \subseteq (I^*(L_f)I^*(L_A))_{I^*(L_B)} = I^*(L_C) = \phi$, implying $I^*(L_f)I^*(L_A) = \phi$ which implies $I^*(L_A) = \phi$ and $\overline{A} \subseteq A \times L_A = \phi \times \phi$ implies $\overline{A} = \phi$ or $A = \Phi$.

Corollary 6.8: *For any 1-p ivf-map* $F: X \to Y$ *and for any nonempty family* $(A_i)_{i \in I}$ *of ivf-subsets of* X *,* $\cap_{i \in I} FA_i = \Phi$ *implies* $\cap_{i \in I} A_i = \Phi$.

Proof: It follows from the above Lemma and 6.4.9.

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