ISSN No. 0976-5697

Volume 3, No. 5, Sept-Oct 2012

International Journal Of Advanced Research In Computer Science

RESEARCH PAPER

Available Online at www.ijarcs.info

Necessary and sufficient condition for Maximal uniquely Hamiltonian graph

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Abstract: A graph is called a maximal uniquely Hamiltonian graph if it has the maximum number of edges among the graphs with the same number of vertices and exact one Hamiltonian cycle. In this paper we present a necessary and sufficient condition for Maximal uniquely Hamiltonian graph and propose a polynomial time algorithm to recognize the maximal uniquely Hamiltonian graph.

Keywords: Hamiltonian cycle, Maximal uniquely Hamiltonian graph, Sheehan graph, Split Hamiltonian graph, 1-tough graph.

I. INTRODUCTION

A. *Definitions and propositions:*

Let $G=(V,E)$ be an undirected and single graph on n vertices, where V be the vertex set and E be edge set of G. We use $|V|$ and $|E|$ to denote the number of vertices and the number edges of G, respectively.

In G, the degree of vertex v is denoted by $d(v)$. The edge of two vertices u and v is denoted by (u,v) or uv, we also say, u and v are end vertices of edge (u,v). For $v\in V$, the set of vertices adjacent to v is denoted by $\Gamma(v)$. For $S\subset V$, $\Gamma(S)$ implies the set of vertices adjacent to vertices in S. A vertex of degree $n-1$ is called a *total vertex* (or complete vertex). The complete graph on n vertices is denoted by K_n.

Two graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are called *isomorphic* if there is a bijection $f:V_1\to V_2$ such that $(u,v)\in E_1$ if and only if $(f(u), f(v))\in E_2$. The graph $H=(W,F)$ is called a *subgraph* of G if $W\subseteq V$ and $F\subseteq E$. Let v is a vertex of G, we use $G-v$ to denote the subgraph which obtained by deleting v from G. Similarly, if B is a set of vertices of G, graph $G-B$ is a subgraph of G whose obtained by deleting B from G. A set of vertices $A\subseteq V$ in a graph G is called *independent* if no two vertices in this set are adjacent, and it is called *clique* if every pair of vertices in this set are adjacent in G. The number of connected components of a graph G is denote by $\omega(G)$. A set $S\subseteq V$ is called *cutset* of G if $\omega(G-S)>1$ (Note that $S=\varnothing$ is cutset iff G is disconnected). A graph G is called *split* if its vertex set V can be partitioned into an independent set I and a clique A, and denoted by $G=(A,I,E)$.

The *Toughness* $t(G)$ of graph G, was defined by Chvátal [1] in the following way: Toughness of a complete graph is infinity, $t(K_n)=\infty$. If G is not complete, then

$$t(G):=\min\{|S|/\omega(G-S):\omega(G-S)>1\}.$$

A graph G is said to be *t-tough* if $t(G)\ge t$ holds, i.e. $|S|\ge t\omega(G-S)$ for every cutset S of G. A cycle of G is called *Hamiltonian cycle* if it contains all vertices of G exactly once. A graph is called *Hamiltonian graph* if it contains a Hamiltonian cycle.

Bondy and Chvátal [1] proved the following result.

Theorem 1 (Bondy-Chvátal). *If G is not-1-tough then G is non-Hamiltonian.*

A finite sequence $d_1,d_2,..., d_n$ of non-negative integers is called the *degree sequence* of a graph G if the vertices can be labeled $v_1,v_2,...,v_n$ and $d(v_i)=d_i$ ($i=\overline{1,n}$). The following result has been proved by Chvátal [3, 11].

Theorem 2 (Chvátal). *Let G be a graph with degree sequence $d_1\le d_2\le ...\le d_n$. If*

$$\forall i, 1\le i<n/2,\ d_i\le i<n/2 \Rightarrow d_{n-i}\ge n-i,$$

then G is Hamiltonian.

The following result of C. T. Hoang [9] is stronger than Theorem 2 of Chvátal.

Theorem 3 (Hoang). *Let G be a graph with degree sequence $d_1\le d_2\le ...\le d_n$ satisfying*

(P): $\forall i, 1\le i<n/2,\ d_i\le i \Rightarrow d_{n-i+1}\ge n-i$.

Then, if G is 1-tough then G is Hamiltonian.

The most famous criterion for degree sequence of graph is due to Erdös and Gallai [4].

Theorem 4 (Erdös-Gallai). *Let $d_1\ge d_2\ge ...\ge d_n>0$ be integers. Then, they are the degree sequence of a graph if only if*

(i) $\sum_{i=1}^{n} d_i$ *is even,*

(ii) For all $k=1,2,...,n-1$,

$$\sum_{i=1}^{k} d_i\le k(k-1)+\sum_{i=k+1}^{n}\min\{k,\ d_i\}.$$

The following result has been proved by Havel - Simeone [6].

Theorem 5 (Havel-Simeone). *Let G be a graph with degree sequence $d_1 \geq d_2 \geq ... \geq d_n > 0$. Then,*

$$G \text{ is split if only if } \sum_{i=1}^{m} d_i = m(m-1) + \sum_{i=m+1}^{n} d_i ,$$

where $m = \max\{k : d_k \geq k-1\}$.

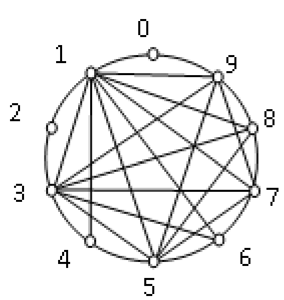
Computing the toughness of a graph G, in general, is a NP-hard problem. In 1980, Burkard and Hammer [2] proposed a necessary condition but not sufficient condition for split Hamiltonian graphs $G = (I, A, E)$ in which $|I| < |A|$. In 2005, Ngo D. Tan and Le H. X. [13] also proved the existence of a Hamiltonian cycle in a split graph such that $5 \neq |I| < |A|$ and $\delta(G) \geq |I| - 3$. In 1996, Kratsch et al. [9] proved that, every $3/2-$tough split is Hamiltonian. Moreover, Gerhard (1998, [5]) proved the following result.

Theorem 6 (Gerhard). *Toughness of a split graph can be computed in polynomial time.*

B. Maximal uniquely Hamiltonian graph

A graph is called *maximal uniquely Hamiltonian graph* (MUHG) if it has the maximum number of edges among the graphs with the same number of vertices and having exactly one Hamiltonian cycle. According to Sheehan [12], MUHG on $n \geq 4$ vertices has exact $[n^2/4]+1$ edges. Sheehan also proposed an algorithm to construct the maximal uniquely Hamiltonian graph as follows.

Algorithm-1 (Sheehan). Firstly, on cycle C the vertices are numbered by the clockwise direction opposite are $0, 1, 2, ..., n-1$ (throughout this paper, integers are taken modulo n). Next, add into C all chords of the form (i, j) with i be odd number and $i < j$.



Figue 1. The Sheehan graph on 10 vertices.

It is easy to verify that, the graph of Algorithm-1 has exactly one Hamiltonian cycle and $[n^2/4]+1$ edges. This graph is also called *Sheehan graph*. Fig. 1 illustrates the Sheehan graph on 10 vertices.

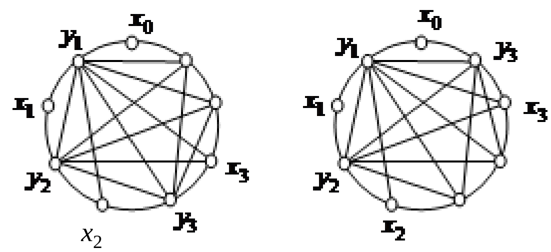
In [7, 8] we proved that, for each $n \geq 7$ there are $2^{[(n-7)/2]}$ MUHG on n vertices whose pairwise are not isomorphic. We also proposed an algorithm to construct $2^{[(n-7)/2]}$ MUHG as follow.

Let $C = (v_0, v_1, ..., v_{n-1})$ be a Hamiltonian cycle in G, where the vertices are numbered by the clockwise direction opposite. We say, distance between two vertices u *and* v on the cycle C, denoted by $d_C(u,v)$, is the length of the shortest path from u to v along the cycle C. For example, for $i < j$, $d_C(v_i, v_j) = \min\{j-i, n+i-j\}$.

Algorithm 2 ([7]):

Firstly, on cycle C with n vertices, for each $i = 1, 2, \cdots, [n/2]-1$, we construct sets X_i and Y_i as follow. Choose x_0 be an any vertex, and set $X_1 := \{x_0\}$, $Y_1 := \varnothing$. Suppose X_i and Y_i have been defined, we define vertex $x_i \in X_i$ so that there exists vertex $x' \in X_i$, $d_C(x_i, x') = 2$, and vertex y_i is adjacent to X_i and X' on C. Set $X_{i+1} = X_i \cup \{x_i\}$, $Y_{i+1} = Y_i \cup \{y_i\}$. Finally, added to the cycle C of edges that connect vertices y_i ($i = 1, 2, ..., [n/2]-1$) with the vertices that do not belong to $X_i \cup Y_i$.

Figue 2 illustrates the two MUHG on 9 vertices are not isomorphic.



Figue 2. The two MUHG on 9 vertices.

II. RESULTS

From Sheehan's Algorithm-1, we can define the degree sequence of MUHG as follows.

Lemma 1. *The increasing degree sequence of MUHG on $n \geq 4$ vertices is defined by*

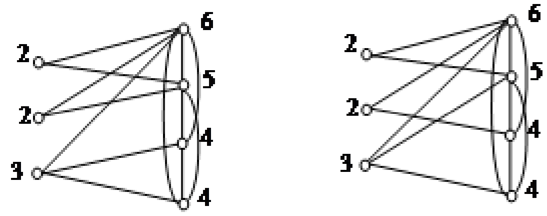
$$d_i = \begin{cases} 2, & i = 1, \\ i, & 2 \leq i \leq [n/2], \\ [n/2]+1, & [n/2]+1 \leq i \leq \lceil n/2 \rceil + 1, \\ i-1, & \lceil n/2 \rceil + 2 \leq i \leq n. \end{cases} \qquad (1)$$

By Theorem 5, it is not difficult to show that, MUHG is split (A, I, E), where A is a clique of $|A| = m = [n/2]+1$ vertices of degree at least $[n/2]$, and I is an independent set of $n-m$ remainder vertices.

It can be seen from Theorem 5, we obtain the following result.

Theorem 7. *Let G be a graph on n vertices with degree sequence satisfying* (1), *then G is split graph.*

Moreover, it is easy to verify that, the degree sequence of a MUHG satisfies predicate (P) of Theorem 3. Note that, let G be a graph on n vertices with degree sequence satisfying (1), then we can not sure that G is Hamiltonian graph. For example, in Figue 3 we have two graphs on 7 vertices with their degree sequence are 2, 2, 3, 4, 4, 5, 6. The first graph is not-1-tough (it's cutset contains two vertices of degree 5 and 6) and so it is non-Hamiltonian (by Theorem 1). The second graph is MUHG.



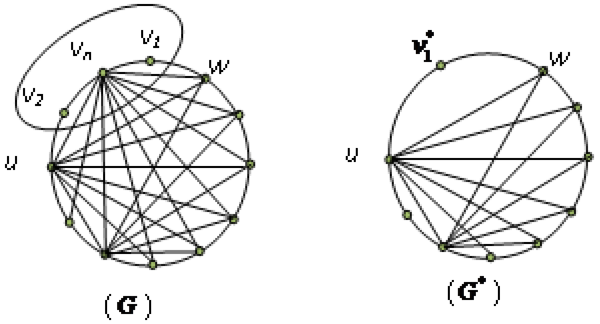
Figue 3. Split non-Hamiltonian and MUHG 7 vertices.

Now let us prove the following result for necessary and sufficient condition of MUHG**.**

Theorem 8. *Graph G is MUHG if only if G with degree sequence satisfying (1) and G is 1-tough graph.*

Proof. Clearly, if G is MUHG then its degree sequence satisfies (1) and G is 1-tough graph.

Otherwise, assume that $G=(V, E)$ be a graph with degree sequence satisfying (1), where $V=\{v_1, v_2, ..., v_n\}$, $d(v_i)=d_i$, $i=1,2,...,n$, and G is 1-tough. Since the degree sequence of G also satisfies predicate (P) and Theorem 3, so G is Hamiltonian. In fact, to complete the proof of Theorem 8, we will show that G has exacty one Hamiltonian cycle. We prove by induction on the number of vertices.



Figue 4. Graphs illustrates the proof of Theorem 7.

It can be seen easily that, Theorem 8 is true for $n=3$ and $n=4$. Suppose that, Theorem 8 is true for every $k<n$: graph G on k vertices with degree sequence satisfying (1) and G is 1-tough, then G is MUHG. Now, we show that Theorem 8 is also true for $n \geq 5$.

Suppose otherwise, graph G has two any Hamiltonian cycles C_1, C_2. By v_n is a total vertex, two vertices v_1, v_2 (degree 2) are non-adjacent (by G be split graph). Thus, there exists two vertices u and w such that v_1 adjacent to u and v_2 adjacent to w (note that $u \neq w$, since otherwise, G is not 1-tough with cutset $S=\{v_n, w\}$ and $\omega(G-S) \geq 3 > |S|$, a contradiction). Moreover, since v_1, v_2 are adjacent to v_n, so two edges (v_1, v_n) and (v_n, v_2) belong to cycles C_1 and C_2, and the remainder edges of v_n are only chords of these Hamiltonian cycles (see Figue 4).

Consider the new graph G^* (see Figue 4) which be obtained from G by replacing three vertices v_1, v_2, v_n with a vertex v_1^* and two edges (v_1^*, u), (v_1^*, w). It is easy to verify that, G^* has $m=n-2$ vertices, $[n^2/4]+1-(n-1)=[(n-2)^2/4]+1 = [m^2/4]+1$ edges, and its degree sequence with m vertices satisfying Lemma 1, and so, it also satisfies predicate (*P*). By the induction hypothesis, G^* has exactly one Hamiltonian cycle, i.e.

$$C_1 - \{v_1, v_2, v_n\} \cup v_1^* = C_2 - \{v_1, v_2, v_n\} \cup v_1^*$$

and it is easy to see that $C_1 \equiv C_2$. We can conclude that graph G has exactly one Hamiltonian cycle C. □

From theorems 6, 7 and 8 we obtain the following result.

Theorem 9. *Recognizing the maximal uniquely Hamiltonian graph can be computed in polynomial time.*

By the proof of Theorem 8, we propose an algorithm for recognizing MUHG in $O(n^2)$ time.

Algorithm-3.

Step 1. If $G = K_3$ or $G = K_4^-$ (by removing one edge from K_4) then G is MUHG; End.

Step 2. If G has not a total vertex or two vertices of degree 2 then G is non-MUHG; End.

Step 3. Let v_n be a total vertex and v_1, v_2 are vertices of degree 2 of G. If $\Gamma(\{v_1, v_2\}) \leq 2$ then G is non-MUHG and End.

Step 4. Let u be a neighbour of v_1 and v be a neighbour of v_2, where $u \neq v_n \neq v$. We define G^* from G by replacing thee vertices v_1, v_2, v_n with a vertex v_1^* and two edges (v_1^*, u), (v_1^*, v). Set $G := G^*$. Return *Step 1.*

III. ACKNOWLEDGMENTS:

First author's research was funded in part by a grant from the University Nha Trang - Vietnam. The second's author's research was supported by the Vietnam National Foundation for Science and Technology Development (NAFOSTED).

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