SOME FIXED POINT THEOREMS IN FUZZY PROBABILISTIC SPACE FOR EXPANSION MAPPING

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Abstract: We have proved some fixed point theorems in Fuzzy Probabilistic Space for expansion mappings.

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1. INTRODUCTION

Menger [3] in 1942 introduced the notation of the probabilistic metric space. The probabilistic generalization of metric space appears to be well adopted for the investigation of physical quantities and physiological thresholds. Schweizer and Sklar [4] studied this concept, subsequently an important development of Menger space theory was due to Sehgal and Bharucha-Reid [5]. So many recent works have been done in fuzzy and menger space [1], [2] and [7]. Fuzzy probabilistic metric space is used by R. Shrivastav, V. Patel and V. B. Dhasgat [6], in their recent work in 2012 in this paper we have proved some fixed point results for fuzzy probabilistic space.

2. PRELIMINARIES

Definition 2.1 A fuzzy probabilistic metric space (FPM space) is an ordered pair consisting of a nonempty set and a mapping from the sets into the collections of all fuzzy distribution functions for all . For we denote the fuzzy distribution function by and is the value of at in . The functions are assumed to satisfy the following conditions:

FPM (1) iff
FPM (2) in
FPM (3) in
FPM (4) If and and

Definition 2.2 A commutative, associative and non-decreasing mapping is a norm if and only if and for

Definition 2.3 A Fuzzy Menger Space is a triplet where is a FPM-space, is a norm and the generalized triangle inequality holds for all in and

The concept of neighborhoods in Fuzzy Menger space is introduced as

Definition 2.4 Let be a fuzzy Menger space. If , and then - neighborhood of called is defined by

An -topology in is the topology induced by the family of neighborhood.

Remark: If is continuous, then Fuzzy Menger space is a Hausdorff space in - topology.
Let be a complete fuzzy menger space and Then is called a bounded set if

Definition 2.5 A sequence in is said to be convergent to a point in if for every and , there exists an integer such that
or equivalently $F_\alpha\ (x_n, x; \varepsilon) > 1 - \lambda$ for all $n \geq N$ and $\alpha \in [0,1]$.

**Definition 2.6** A sequence $\{x_n\}$ in $(X, F_\alpha, t)$ is said to be a Cauchy sequence if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $N = N(\varepsilon, \lambda)$ such that

$$
F_\alpha(\varepsilon, x_m; x_n; \varepsilon) > 1 - \lambda \forall \ n, \ m \geq N.
$$

**Definition 2.7** A Fuzzy Menger space $(X, F_\alpha, t)$ with the continuous t-norm is said to be complete if every Cauchy sequence in $X$ converges to a point in $X$ for all $\alpha \in [0,1]$.

Let $(X, F_\alpha, t)$ be a complete fuzzy Menger space with the following condition

**(FMS-6)**

$$
\lim_{t \to \infty} F_\alpha(x, y, t) = 1,
$$

$\forall x, y \in X$.

Following lemmas is selected from [7] in fuzzy Menger space.

**Lemma 2.1.** Let $\{x_n\}$ be a sequence in a Menger space $(X, F_\alpha, \ast)$ with continuous t-norm $\ast$ and $t \ast t \geq t$. If there exists a constant $k \in (0, 1)$ such that

$$
F_\alpha(x_n, x_{n+1})(k) \geq F_\alpha(x_{n-1}, x_n)(t) \text{ for all } t > 0 \text{ and } n = 1, 2, \ldots,
$$

then $\{x_n\}$ is a Cauchy sequence in $X$.

**Lemma 2.2.** Let $(X, F_\alpha, \ast)$ be a Menger space. If there exists $k \in (0, 1)$ such that

$$
F_\alpha(x, y)(k) \geq F_\alpha(x, y)(t) \quad \text{for all } x, y \in X \text{ and } t > 0,
$$

then $x = y$.

### 3. MAIN RESULTS

**THEOREM 3.1:** Let $X$ be a complete Fuzzy probabilistic metric space and $T$ be a self-map of $X$. The mapping $T$ satisfying the condition;

$$
F_\alpha(Tx,Ty,kt) \geq \min \left\{ F_\alpha(x,T(x),t)F_\alpha(Ty,y,T(y),t), F_\alpha(x,T(x),t)F_\alpha(y,T(y),t), F_\alpha(x,y,t) \right\}
$$

for all $x, y \in X$ with $x \neq y$ and $T$ is onto. Then $T$ has a fixed point in $X$.

**PROOF:** Let $x_0 \in X$ since $T$ is onto there is an element $x_1$ satisfying $x_1 \in T^{-1}(x_0)$. By the same way we can choose, $x_n \in T^{-1}x_{n-1}$, where $(n = 2, 3, 4, - - -)$. If $x_{m-1} = x_m$ for some $m$, then $x_m$ is a fixed point of $T$. Without loss of generality we can suppose $x_{n-1} \neq x_n$ for every $n$. So

$$
F_\alpha(Tx_n, T(x_{n-1}), kt) = F_\alpha(T(x_{n-1}), x_n, t) \geq
$$

$$
\int F_\alpha(Tx_n, T(x_{n-1}), t) \geq [F_\alpha(x_n, T(x_{n-1}), t)F_\alpha(T(x_{n-1}), T(x_{n+1}), t)F_\alpha(x_{n+1}, T(x_{n+1}), t) + F_\alpha(x_{n+1}, T(x_{n+1}), t)F_\alpha(x_{n+1}, T(x_{n+1}), t)]
$$

Therefore by well known way $\{x_n\}$ is a Cauchy square in $X$. Since $X$ is complete $\{x_n\}$ converges to $x$, for some $x \in X$. Since $T$ is onto there exists $y \in X$ such that $y \in T^{-1}(x)$ and for infinitely many $n, x_n \neq x$, for such $n$

$F_\alpha(x_n, x) = F_\alpha(T(x_{n+1}), T(y), kt) \geq 0$ \text{ since } F_\alpha(y, T(y), t) \geq 0.

So, in both cases we get $x \neq y$. Thus $T$ has a fixed point in $X$.

This completes the proof.

**THEOREM 3.2:** Let $X$ be a complete fuzzy probabilistic metric space and $T$ be a self-map of $X$. The mapping $F$ satisfying the condition;

$$
F_\alpha(Tx, Ty, t) \geq \int F_\alpha(x, T(x), t)F_\alpha(y, T(y), t)F_\alpha(y, T(x), t) + F_\alpha(x, T(x), t)F_\alpha(y, T(y), t)F_\alpha(x, T(y), t)
$$

for all $x, y \in X$ with $x \neq y$ and $T$ is onto. Then $T$ has a fixed point in $X$.

**PROOF:** Let $x_0 \in X$ since $T$ is onto there is an element $x_1$ satisfying $x_1 \in T^{-1}(x_0)$. By the same way we can choose, $x_n \in T^{-1}x_{n-1}$, where $(n = 2, 3, 4, - - -)$. If $x_{m-1} = x_m$ for some $m$, then $x_m$ is a fixed point of $T$. Without loss of generality we can suppose $x_{n-1} = x_n$ for every $n$. So

$$
F_\alpha(Tx_n, T(x_{n+1}), t) \geq
$$

$$
\int F_\alpha(x_{n+1}, T(x_{n+1}), t)F_\alpha(x_{n+1}, T(x_{n+1}), t)F_\alpha(x_{n+1}, T(x_{n+1}), t) + F_\alpha(x_{n+1}, T(x_{n+1}), t)F_\alpha(x_{n+1}, T(x_{n+1}), t)F_\alpha(x_{n+1}, T(x_{n+1}), t)
$$

for all $x, y \in X$ with $x \neq y$ and $T$ is onto. Then $T$ has a fixed point in $X$. 

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Then for all \( x, y \in X \) we can write

\[
F_a(x_n, x_{n+1}, t) \geq \begin{cases}
F_a(x_n, T(x_n), t)F_a(x_{n+1}, T(x_n), t) + F_a(x_n, T(x_{n+1}), t)F_a(x_{n+1}, T(x_n), t) & \text{if } n \text{ is even} \\
F_a(x_n, x_{n+1}, t) & \text{if } n \text{ is odd}
\end{cases}
\]

\[
\varnothing = \{ F_a(x_n, x_{n+1}, t) \}
\]

\[
F_a(x_n, x_{n+1}, t) \geq \varnothing \}
\]

Therefore \( \{ x_n \} \) is a Cauchy sequence in \( X \) and \( X \) is complete therefore \( \{ x_n \} \) converges to \( x \) for some \( x \in X \). By continuity of \( T \) we can write

\[
T(x_n) \to T(x), \quad n \to \infty
\]

Hence \( T(x) = x \)

This completes the proof.

**THEOREM 3.3:** Let \( X \) be a complete fuzzy probabilistic metric space and \( T \) be a self-map of \( X \). The mapping \( T \) satisfying the condition;

\[
F_a(Tx, Ty, t) \geq \min \left[ \begin{array}{c}
F_a(x, y, t)F_a(y, T(y), t), F_a(x, T(y), t)F_a(y, T(x), t)
\end{array} \right]
\]

for all \( x, y \in X \) with \( x \neq y \) and \( T \) is onto, there exists a point \( w \) in \( X \) such that:

\[
S(w) = \sup \{ S(x) : S(x) = M(x; T(x), t), x \in X \}
\]

Then \( T \) has a fixed point in \( X \).

**PROOF:** Let \( w \neq T(w) \), otherwise \( w \) is a fixed point of \( T \).

Put \( x = w \) and \( y = T(w) \)

\[
F_a(T(w), T(w), t) \geq \min \left[ \begin{array}{c}
F_a(w, T(w), t)F_a(T(w), T(w), t), F_a(T(w), T(w), t)F_a(T(w), T(w), t)
\end{array} \right]
\]

\[
F_a(T(w), T(w), t) \geq F_a(T(w), T(w), t)
\]

which is not possible.

So

\[
F_a[T(w), T^2(w), t] \geq F_a[w, T(w), t] \quad \text{... (1.3.2)}
\]

Similarly on putting \( x = T(w) \) and \( y = w \), we get

\[
F_a[T^2(w), T(w), t] \geq \min \left[ \begin{array}{c}
F_a(T(w), T(w), t)F_a(T(w), T(w), t), F_a(T(w), T(w), t)F_a(T(w), T(w), t)
\end{array} \right]
\]

\[
F_a[T(w), T(w), t] \quad \text{or}
\]

\[
F_a[w, T(w), t] \quad \text{... (1.3.2)}
\]

By (1.3.1) and (1.3.2)

\[
F_a[T(w), T^2(w), t] \geq F_a[w, T(w), t]
\]

\[
F_a[T(w), T^2(w), t] \geq F_a[w, T(w), t] = S(w)
\]

This implies that

\[
S(T(w)) > S(w)
\]

giving a contraction.

Hence we must have \( T(w) = w \), that is \( w \) is a fixed point of \( T \) in \( X \).

**REFERENCES**


