SOME FUNCTIONS DEFINED BY $D_\beta$-CLOSED SET

Purushottam Jha  
Department of Mathematics  
Govt. P. G. College, Naraynapur, Chhattisgarh, India-494661

Manisha Shrivastava  
Department of Mathematics  
Govt. J. Y. Chhattisgarh College, Raipur, Chhattisgarh, India-492001

Abstract: In [16] present authors introduced and studied a new class of generalized closed sets called $D_\beta$-closed sets, $D_\beta$-open sets, $D_\beta$-continuous and $D_\beta$-irresolute functions in topological spaces. In this paper we introduce and investigate new class of open and closed functions called $D_\beta$-open and $D_\beta$-closed functions by using $D_\beta$-open and $D_\beta$-closed sets and explore their fundamental properties. The concept of $D_\beta$-closed graph, strongly $D_\beta$-closed graph and some weak separation axioms are also extend by using the notion of $D_\beta$-open sets.

2010 Mathematics Subject Classification: Primary: 54A05, 54C05, 54C08; Secondary: 54C10, 54D10, 54C99.

Keyword: $D_\beta$-closed, $D_\beta$-open, $D_\beta$-continuous, $D_\beta$-irresolute, $D_\beta$ - $T_0$, $D_\beta$ - $T_1$, $D_\beta$ - $T_2$, $D_\beta$-closed graph, Strongly $D_\beta$-closed graph.

1. INTRODUCTION

Monsef et al. [1] devised and investigated a new notion of open set which is one of the member of class of basic open sets called $\beta$-open set. This kind of set discussed by Andrijevic [3] under the name, semi-preopen sets. The concept of generalized closed (briefly g-closed) sets in a topological space and a class of topological spaces called $T_{3\frac{1}{2}}$-space was introduced by Levine [18] and these sets were further considered by Dunham and Levine [11]. Dunham (See also [10]) continued the study of $T_{3\frac{1}{2}}$-spaces.

In 1982 Dunham [12] derived a new closure operator $C^*$ by using g-closed sets in such a way that for any topological space $(X, \tau)$, $C^*(E) = \cap \{A: E \subseteq A \in D\}$, where $D = \{A: A \subseteq X, A$ is g-closed $\}$ and he also proved that $C^*$ is a Kuratowski closure operator in X. Dunham [11] also proved that $(X, \tau)$ is always a $T_0$-space and by improving the result, he established that $(X, \tau^*)$ is $T_{3\frac{1}{2}}$-space, for any topological space $(X, \tau)$ . Munshi et al. [25] introduced the notion of g-continuous functions. Balchandran et al. [4] also studied the notion of g-continuity. Malghan [21] introduced the generalized closed maps and discussed its fundamental properties. $\beta$-open ( $\beta$-closed) functions defined and discussed by Monsef et al. [1]. Mahmoud et al. [20] initiated and studied the lower separation axioms via $\beta$-open sets. Caldas (see [8], [5], [6]) discussed some properties of functions with strongly $\alpha$-closed graphs by utilizing $\alpha$-open sets and the $\alpha$-closure operator. Long [19] investigated the required condition for a graph $G(f)$ to be a closed subset of the product space $X \times Y$. Herrington et al. [14] introduced the concept of strongly closed graph. Noiri [27] gave some characterization of strongly closed graph. Sayed et al. [29] introduced $D_\alpha$-closed sets and in topological spaces by using the generalized closure operator $C^*$ due to Dunham [12] and also established $D_\alpha$-continuous, $D_\alpha$-open and $D_\alpha$-closed graph and strongly $D_\alpha$-closed graph functions in topological spaces. In this paper, in Section 1, we give some basic definitions and results which will use in the sequel. In Section 2, we define $D_\beta$-open and $D_\beta$-closed functions and characterize its fundamental properties. In Section 3, we devise and elaborate some lower separation axioms using $D_\beta$-open sets. In Section 4, we discuss the concepts of $D_\beta$-closed graphs and strongly $D_\beta$-closed graphs.

1.1 Preliminaries. Throughout this paper $(X, \tau)$ will always denote a topological space on which no separation axioms are assumed, unless explicitly stated. If $A$ is a subset of the space $(X, \tau)$, $C\ell(A)$ and $Int(A)$ denote the closure and the interior of $A$ respectively. Here we recall the following known definitions and properties.

Definition 1.1. Let $(X, \tau)$ be a topological space. A subset $A$ of the space $X$ is said to be,

(i) preopen [22] if $A \subseteq Int (C\ell(A))$ and preclosed if $C\ell (Int(A)) \subseteq A$.

(ii) semiopen [17] if $A \subseteq C\ell (Int(A))$ and semiclosed if $Int (C\ell(A)) \subseteq A$.
(iii) \( \alpha \)-open [20] if \( A \subseteq \text{Int}(C(\text{Int}(A))) \) and \( \alpha \)-closed if \( C(\text{Int}(C(\text{Int}(A)))) \subseteq A \).

(iv) \( \beta \)-open [1] if \( A \subseteq C(\text{Int}(C(\text{Int}(A)))) \) and \( \beta \)-closed if \( \text{Int}(C(\text{Int}(A))) \subseteq A \).

(v) generalized closed (briefly g-closed) [16] if \( C(\text{Int}(A)) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open in \( X \) and generalized open (briefly g-open) if \( X \setminus A \) is g-closed.

(vi) pre\(^*\)-closed set [30] if \( C^*\text{Int}(A) \subseteq A \) and pre\(^*\)-open set if \( A \subseteq C^*\text{Int}(A) \).

(vii) semi\(^*\)-closed set [28] if \( \text{Int}^*(C^*(\text{Int}(A))) \subseteq A \) and semi\(^*\)-open set [23] if \( A \subseteq C^*\text{Int}^*(\text{Int}(A)) \).

(viii) \( D_\alpha \)-closed [29] if \( C(\text{Int}(C(\text{Int}(A)))) \subseteq A \) and \( D_\alpha \)-open if \( X \setminus A \) is \( D_\alpha \)-closed.

(ix) \( D_\beta \)-closed [16] \( \text{Int}(C(\text{Int}(C(\text{Int}(A)))) \subseteq A \) and \( D_\beta \)-open if \( X \setminus A \) is \( D_\beta \)-closed.

Definition 1.2. A function \( f : (X, \tau) \to (Y, \sigma) \) is said to be,

(i) \( \beta \)-continuous [1] if the inverse image of each open set in \( Y \) is \( \beta \)-open in \( X \).

(ii) g-continuous [4] if the inverse image of each open set in \( Y \) is g-open in \( X \).

(iii) pre\(^*\)-continuous [30] if the inverse image of each open set in \( Y \) is pre\(^*\)-open in \( X \).

(iv) semi\(^*\)-continuous [23] if the inverse image of each open set in \( Y \) is semi\(^*\)-open in \( X \).

(v) \( D_\alpha \)-continuous [29] if the inverse image of each open set in \( Y \) is \( D_\alpha \)-open in \( X \).

(vi) \( D_\beta \)-continuous [16] if the inverse image of each open set in \( Y \) is \( D_\beta \)-open in \( X \).

(vii) \( \beta \)-open [1] (resp. \( \beta \)-closed ) if the image of each open (resp. closed ) set in \( X \) is \( \beta \)-open (resp. \( \beta \)-closed) in \( Y \).

(viii) g-open [21] (resp. g-closed ) if the image of each open (resp. closed ) set in \( X \) is g-open (resp. g-closed) in \( Y \).

(ix) semi\(^*\)-open (resp. semi\(^*\)-closed) [24] if the image of each open (resp. closed) set in \( X \) is semi\(^*\)-open (resp. semi\(^*\)-closed) in \( Y \).

(x) \( D_\alpha \)-open [29] (resp. \( D_\alpha \)-closed) if the image of each open (resp. closed) set in \( X \) is \( D_\alpha \)-open (resp. \( D_\alpha \)-closed) in \( Y \).

Definition 1.3. Let \( f : (X, \tau) \to (Y, \sigma) \) be a function:

(i) the subset \( \{(x, f(x)) | x \in X\} \) of the product space \( (X \times Y) \) is called the graph of \( f \) [19] and is usually denoted by \( \text{G}(f) \).

(ii) has a closed graph [19] if its graph \( \text{G}(f) \) is closed sets in the product space \( X \times Y \).

(iii) has a strongly closed graph [14] if for each point \( (x, y) \) not belongs to \( \text{G}(f) \), there exists open sets \( U \subseteq X \) and \( V \subseteq Y \) containing \( x \) and \( y \), respectively such that \( (U \times \text{C}(V)) \cap \text{G}(f) = \phi \).

(iv) has \( D_\alpha \)-closed graph [29] if for each \( (x, y) \in (X \times Y) \setminus \text{G}(f) \), there exists \( U \subseteq D_\alpha \text{O}(X, x) \) and \( V \subseteq \text{GO}(Y, y) \) such that \( (U \times \text{C}(V)) \cap \text{G}(f) = \phi \).

(v) has a strongly \( D_\alpha \)-closed graph [29] if for each \( (x, y) \in (X \times Y) \setminus \text{G}(f) \), there exists an \( \alpha \)-open set \( U \) in \( X \) and \( V \subseteq \text{O}(Y) \) such that \( (U \times \text{C}(V)) \cap \text{G}(f) = \phi \).

Definition 1.4. A topological space \( (X, \tau) \) is said to be,

(i) \( T_{1/2} \) [18] if every g-closed set is closed.

(ii) \( g \cdot T_0 \) [7] if for any distinct pair of points \( x \) and \( y \) in \( X \), there exists a g-open set \( U \) in \( X \) containing \( x \) but not \( y \) or containing \( y \) but not \( x \).

(iii) \( g \cdot T_1 \) [7] if for any distinct pair of points \( x \) and \( y \) in \( X \), there exists g-open set \( U \) in \( X \) containing \( x \) but not \( y \) and an g-open set \( V \) in \( X \) containing \( y \) but not \( x \).

(iv) \( g \cdot T_2 \) [7] if for any distinct pair of points \( x \) and \( y \) in \( X \), there exists a g-open sets \( U \) and \( V \) in \( X \) containing \( x \) and \( y \), respectively such that \( U \cap V = \phi \).

(v) \( \beta \cdot T_0 \) [20] if for any distinct pair of points \( x \) and \( y \) in \( X \), there exists \( \beta \)-open set \( U \) in \( X \) containing \( x \) but not \( y \) or containing \( y \) but not \( x \).

(vi) \( \beta \cdot T_1 \) [20] (resp. \( D_\alpha \cdot T_1 \) [24]) if for any distinct pair of points \( x \) and \( y \) in \( X \), there exists \( D_\alpha \)-open (resp. \( D_\alpha \)-closed) set \( U \) in \( X \) containing \( x \) but not \( y \) and an \( \beta \)-open (resp. \( D_\alpha \)-open) set \( V \) in \( X \) containing \( y \) but not \( x \).

(vii) \( \beta \cdot T_2 \) [20] ( resp. \( D_\alpha \cdot T_2 \) [29] ) if for any distinct pair of points \( x \) and \( y \) in \( X \), there exists \( \beta \)-open (resp. \( D_\alpha \)-open) sets \( U \) and \( V \) in \( X \) containing \( x \) and \( y \), respectively such that \( U \cap V = \phi \).

The intersection of all g-closed sets containing \( A \) [11] is called the g-closure of \( A \) and denoted by \( \text{C}(A) \) and the g-interior of \( A \) [12] is the union of all g-open sets contained in \( A \) and is denoted by \( \text{Int}^*(A) \). The family of all \( D_\beta \)-closed (resp. closed , \( D_\alpha \)-closed, g-closed, \( \beta \)-closed, semi\(^*\)-closed) sets of \( X \) denoted by \( D_\beta \text{C}(X) \) (resp., \( \text{C}(X) \), \( D_\beta D_\alpha \text{C}(X), \text{GC}(X), \beta \text{C}(X), \text{semi}^*\text{C}(X) \)). The family of all \( D_\beta \)-open (resp. open set of, \( D_\alpha \)-open, g-open, \( \beta \)-open, semi\(^*\)-open) sets of \( X \) denoted by \( D_\beta \text{O}(X) \) (resp. \( \text{O}(X), D_\alpha \text{O}(X), \text{GO}(X), \beta \text{O}(X), \text{semi}^*\text{O}(X) \)).
Lemma 1.5. [19] A function $f : (X, \tau) \to (Y, \sigma)$ has a closed graph if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exists $U \in O(X, x)$ and $V \in O(Y, y)$ such that $f(U) \cap V = \emptyset$.

Lemma 1.6. [27] The graph $G(f)$ is strongly closed if and only if for each point $(x, y) \in G(f)$, there exists open sets $U \subset X$ and $V \subset Y$ containing $x$ and $y$, respectively, such that $f(U) \cap C(V) = \emptyset$.

2. $D_\beta$-OPEN AND $D_\beta$-CLOSED FUNCTIONS

Definition 2.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $D_\beta$-open (resp. $D_\beta$-closed) if the image of each open (resp. closed) set in $X$ is $D_\beta$-open (resp. $D_\beta$-closed) in $Y$.

Theorem 2.2. (i) Every $\beta$-open function is $D_\beta$-open.

(ii) Every $g$-open function is $D_\beta$-open.

(iii) Every semi*-open function is $D_\beta$-open.

(iv) Every $D_\alpha$-open function is $D_\beta$-open.

Proof. (i) The proof follows from the definition and from the Theorem 2.3 of [16] that every $g$-open set is $D_\beta$-open.

(ii) The proof follows from the definition and from the Theorem 2.3 of [16] that every semi*-open set is $D_\beta$-open.

(iii) The proof follows from the definition and from the Theorem 2.3 of [16] that every $\beta$-open set is $D_\beta$-open.

(iv) The proof follows from the definition and from the Theorem 2.3 of [16] that every $D_\alpha$-open set is $D_\beta$-open.

Remark. (i) $D_\beta$-open function need not be $\beta$-open. (see the Example 2.3 below)

(ii) $D_\beta$-open function need not be $g$-open. (see the Example 2.4 below)

(iii) $D_\beta$-open function need not be semi*-open (see the Example 2.4 below)

(iv) $D_\beta$-open function need not be $D_\alpha$-open. (see the Example 2.5 below)

Example 2.3. Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{b, c\}, \{c\}\}$, then $(X, \tau)$ be a topological space. Let $Y = \{1, 2, 3\}$ and $\sigma = \{Y, \phi, \{2, 3\}, \{3\}\}$, then $(Y, \tau)$ be a topological space.

$C(Y) = \{Y, \phi, \{1\}, \{1, 2\}\}$,
$\beta C(Y) = \{Y, \phi, \{1, 2\}, \{1\}, \{2\}\}$,
$\beta O(Y) = \{Y, \phi, \{2, 3\}, \{3\}, \{1, 3\}\}$,
$D_\beta O(C(Y)) = \{Y, \phi, \{1\}, \{1, 2\}, \{2\}, \{1, 3\}, \{3\}\}$.

Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined by $f(a) = 3, f(b) = 1, f(c) = 2$ is $D_\beta$-open function, since the image of each open set is $D_\beta$-open in $Y$. But $f$ is not a $\beta$-open function, since $f(\{b, c\}) = \{1, 2\}$, which is not $\beta$-open in $Y$.

Example 2.4. Let $X = \{a, b, c\}$ be any set and $\tau = \{X, \phi, \{a, b\}\}$, then $(X, \tau)$ be a topological space. Let $Y = \{x, y, z\}$ and $\sigma = \{Y, \phi, \{y, z\}, \{z\}\}$, then $(Y, \sigma)$ be a topological space.

$C(Y) = \{Y, \phi, \{x, y\}\}$,
$GC(Y) = \{Y, \phi, \{x, y, z\}\}$,
$GO(Y) = \{Y, \phi, \{y, z\}, \{z\}\}$,
$D_\beta C(Y) = \{Y, \phi, \{x, y, z\}, \{y, z\}, \{x, z\}, \{x, y\}\}$,
$S' O(Y) = \{Y, \phi, \{y, z\}, \{z\}, \{x, z\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the function defined by $f(a) = x, f(b) = y, f(c) = z$ is $D_\beta$-open function, since $f$ image of each open set is $D_\beta$-open in $Y$. But $f$ is not g-open, since $f(\{a, b\}) = \{x, y\}$, which is not g-open in $Y$ and not semi*-open in $Y$.

Example 2.5. Let $X = \{x, y, z\}$ and $\tau = \{X, \phi, \{y, z\}, \{z\}\}$, then $(X, \tau)$ be a topological space. Let $Y = \{a, b, c\}$ and $\sigma = \{Y, \phi, \{b, c\}, \{c\}\}$, then $(Y, \sigma)$ be a topological space. Let $C(Y) = \{Y, \phi, \{a\}, \{a, c\}, \{a, b\}\}$,
$GC(Y) = \{Y, \phi, \{a, c\}, \{a, b\}\}$,
$GO(Y) = \{Y, \phi, \{b, c\}, \{b\}, \{c\}\}$,
$D_\alpha C(Y) = \{Y, \phi, \{a\}, \{a, c\}, \{a, b\}\}$,
$D_\alpha O(Y) = \{Y, \phi, \{b, c\}, \{b\}, \{c\}\}$.

Let function $f : (X, \tau) \to (Y, \sigma)$ be the function defined by $f(x) = b, f(y) = a$, and $f(z) = c$ is $D_\alpha$-open function, since the image of each open set in $X$ is $D_\alpha$-open in $Y$, but $f$ is not $D_\alpha$-open.

Since $f(\{y, z\}) = \{a, c\}$, which is not $D_\alpha$-open in $Y$.

Remark. The composition of two $D_\alpha$-open maps need not be $D_\beta$-open in general. This is shown by the following example.

Example 2.6. Let $X = Y = Z = \{a, b, c\}$ be the sets with the topology $\tau = \{X, \phi, \{a, b, d\}, \{b, d\}\}$,
$\sigma = \{Y, \phi, \{a, b\}, \{a, b, c\}, \{b, d\}, \{b\}, \{a, b, d\}\}$ and
$\eta = \{Z, \phi, \{b, c, d\}, \{c, d\}, \{a, b\}, \{b\}, \{a, b, c\}\}$, respectively. Then $(X, \tau)$, $(Y, \sigma)$ and $(Z, \eta)$ be the
topological spaces. We define \( f : (X, \tau) \rightarrow (Y, \sigma) \) as \( f(a) = b, f(b) = c, f(c) = d \) and the map \( g : (Y, \sigma) \rightarrow (Z, \eta) \) as \( g(a) = d, g(b) = c, g(c) = a \) and \( g(d) = a \). Then \( f \) and \( g \) are \( D_\beta \)-open maps, but their composition \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is not \( D_\beta \)-open map, because any open set in \((X, \tau)\) and \( g \circ f(A) = g(f(\{b, d\})) = g(\{a, c\}) = \{a, d\} \), which is not a \( D_\beta \)-closed set in \((Z, \eta)\).

**Theorem 2.7.** If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is a open map and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) is \( D_\beta \)-open map, then their composition \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is \( D_\beta \)-open map.

**Proof.** Let \( G \) be any open set in \((X, \tau)\). Since \( f \) is open map, \( f(G) \) is open in \((Y, \sigma)\). Since \( g \) is \( D_\beta \)-open map, \( g(f(G)) \) is \( D_\beta \)-open set in \((Z, \eta)\). Therefore \( g \circ f(G) = g(f(G)) \) is \( D_\beta \)-open set in \((Z, \eta)\).

**Theorem 2.8.**

(i) Every \( g \)-closed function is \( D_\beta \)-closed.

(ii) Every \( \alpha \)-closed function is \( D_\beta \)-closed.

(iii) Every \( \beta \)-closed function is \( D_\beta \)-closed.

(iv) Every \( D_\beta \)-closed function is \( D_\beta \)-closed.

**Proof.** It is obvious.

**Theorem 2.9.** If the space \( X \) is \( T_{1/2} \), then every \( D_\alpha \)-closed (resp. \( D_\beta \)-closed) set is \( \alpha \)-closed (resp. \( \beta \)-closed).

**Proof.** Let \( A \) be any \( D_\alpha \)-closed (resp. \( D_\beta \)-closed) subset of the space \( X \), then we have \((C\ell^*(\text{Int}(C\ell^*(A)))) \subseteq A\) (resp. \(\text{Int}^*(C\ell^*(\text{Int}^*(A)))) \subseteq A\). Since the space \( X \) is \( T_{1/2} \), every \( g \)-closed set is closed, consequently \( C\ell^*(A) = C\ell(A) \). Therefore, we get \(\text{Int}^*(C\ell(\text{Int}(A)))) \subseteq A\) (resp. \(\text{Int}(C\ell(\text{Int}(A)))) \subseteq A\).

**Theorem 2.10** If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is a \( g \)-closed map, \( g : (Y, \sigma) \rightarrow (Z, \eta) \) is \( D_\beta \)-closed map and \((Y, \sigma)\) be \( T_{1/2} \)-space, then their composition \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is \( D_\beta \)-closed and therefore \( \beta \)-closed.

**Proof.** Let \( A \) be any closed set in \((X, \tau)\). Since map \( f \) is \( g \)-closed, \( f(A) \) is \( g \)-closed in \((Y, \sigma)\). Since \((Y, \sigma)\) is \( T_{1/2} \)-space, \( f(A) \) is closed in \((Y, \sigma)\). Since \( g \) is \( D_\beta \)-closed, \( g(f(A)) \) is \( D_\beta \)-closed set and therefore \( \beta \)-closed set in \((Z, \eta)\) and \( g \circ f(A) = g(f(A)) \). Hence the composition map \( g \circ f \) is \( D_\beta \)-closed and therefore \( \beta \)-closed.

**Theorem 2.11.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be any bijective map, then the following statements are equivalent;

(i) \( f^{-1} \) is \( D_\beta \)-continuous.

(ii) \( f \) is \( D_\beta \)-open map.

(iii) \( f \) is \( D_\beta \)-closed map.

**Proof.**

(i) \( \Rightarrow \) (ii): Let \( U \) be any open set in \( X \). By assumption \((f^{-1})^{-1}(U) = f(U) \) is \( D_\beta \)-open in \( Y \). This shows that \( f \) is \( D_\beta \)-open map.

(ii) \( \Rightarrow \) (iii): Let \( F \) be any closed set in \( X \). Then \( F^c \) is open set in \( X \), therefore by assumption \( f(F^c) = (f(F))^c \) is \( D_\beta \)-open in \( Y \), consequently \( f(F) \) is \( D_\beta \)-closed in \( Y \). Hence the map \( f \) is \( D_\beta \)-closed.

(iii) \( \Rightarrow \) (iv): Let \( F \) be any closed set in \( X \). Then by assumption \( f(F) \) is \( D_\beta \)-closed in \( Y \) and therefore \( f(F) = (f^{-1})^{-1}(F) \) is \( D_\beta \)-closed in \( Y \), therefore \( f^{-1} \) is \( D_\beta \)-continuous.

### 3. LOWER SEPARATION AXIOMS

Separations are very useful concepts in topological spaces. They can be used to define the more restricted classes of topological spaces. The separation axioms exhibit the knowledge about the points and sets that are distinguishable or separated in some weaker sense or some stronger sense. Here we define and study some new types of lower separation axioms, namely \( D_\beta \rightarrow T_n \) for \( n = 0, 1 \) and 2.

**Definition 3.1.** A topological space \((X, \tau)\) is said to be,

(i) \( D_\alpha \rightarrow T_0 \) if for any distinct pair of points \( x \) and \( y \) in \( X \), there exists a \( D_\alpha \)-open set \( U \) in \( X \) containing \( x \) but not \( y \) or containing \( y \) but not \( x \).

(ii) \( D_\beta \rightarrow T_0 \) if for any distinct pair of points \( x \) and \( y \) in \( X \), there exists a \( D_\beta \)-open set \( U \) in \( X \) containing \( x \) but not \( y \) or containing \( y \) but not \( x \).

(iii) \( D_\beta \rightarrow T_1 \) if for any distinct pair of points \( x \) and \( y \) in \( X \), there exists a \( D_\beta \)-open set \( U \) in \( X \) containing \( x \) but not \( y \) and a \( D_\beta \)-open set \( V \) in \( X \) containing \( y \) but not \( x \).

(iv) \( D_\beta \rightarrow T_2 \) if for any distinct pair of points \( x \) and \( y \) in \( X \), there exists a \( D_\beta \)-open set \( U \) in \( X \) containing \( x \) but not \( y \) and \( D_\beta \)-open set \( V \) in \( X \) containing \( y \) but not \( x \).

**Remark.** Every \( D_\beta \rightarrow T_0 \) space need not be \( D_\beta \rightarrow T_1 \).

**Example 3.2.** Let \( X = \{a, b, c\} \) be any set and \( \tau = \{\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\} \), then \((X, \tau)\) be a topological space. 

\[
\text{C}(X) = \{\phi, X, \{c\}, \{a, c\}, \{a\}\}, \\
\text{D}_\beta \text{C}(X) = \{X, \phi, \{c\}, \{a, c\}, \{a\}\}, \\
\text{D}_\beta \text{O}(X) = \{X, \phi, \{a, b\}, \{b\}, \{b, c\}\}.
\]

Thus the space \( X \) is
Dβ - T0 but it is not Dβ - T1. Since for a pair of distinct points a and b, there exists two Dβ-open sets {a, b} and {b} in which a ∈ {a, b} and a ≠ {b} but b ∈ {a, b} and also b ∈ {b}.

Remark. Every Dβ - T1 space need not be Dβ - T2.

Example 3.3. Let X = {a, b, c} be any set and τ = {X, φ, {a, c}, {c}} , then (X, τ) be a topological space.

C(X) = {φ, X, {b}, {a, b}} ,

Dβ C(X) = {X, φ, {b}, {a, b}, {b, c}, {a}, {c}},

Dβ O(X) = {X, φ, {a, c}, {c}, {b, c}, {a}, {a, b}}.

Thus the space X is Dβ - T1, but not Dβ - T2. Since for the distinct pair of points a and b, there are two Dβ-open sets {a, c} and {b, c} such that a ∈ {a, c} and b ∈ {b, c}, but these sets are not disjoint.

Remark. (i) Every Dα - T0 space is Dβ - T0.

(ii) Every Dα - T1 space is Dβ - T1.

(iii) Every Dα - T2 space is Dβ - T2.

Proof. It is obvious.

Theorem 3.5. A space X is Dβ - T0 space if and only if for each pair of distinct points x, y of X, CℓDβ (x) ≠ CℓDβ (y).

Proof. Necessity - Let (X, τ) be any topological space and let x, y be any two points in X with x ≠ y. Since X is Dβ - T0 , then there exists a Dβ-open set A such that x ∈ A but y ∉ A. Therefore x ∉ X \ A and y ∈ (X \ A), where X \ A is Dβ-closed in X. Since CℓDβ (y) is the smallest Dβ-closed set containing {y} and therefore y ∈ CℓDβ (y) and CℓDβ (y) ⊆ X \ A. Therefore x ∉ CℓDβ (y). Hence CℓDβ (x) ≠ CℓDβ (y).

Sufficiency - Suppose for any disjoint pair points p, q of X with CℓDβ (p) ≠ CℓDβ (q) then there exists at least one point r ∈ X such that r ∈ CℓDβ (p) but r ∉ CℓDβ (q). On contrary suppose p ∈ CℓDβ (q) , then CℓDβ (p) ⊆ CℓDβ (q) , consequently r ∉ CℓDβ (q) , which is the contradiction of the fact that r ∉ CℓDβ (q). Hence p ∈ (X \ CℓDβ (q)) and q ∉ (X \ CℓDβ (q)). This implies that X is Dβ - T0.

Theorem 3.6. Let f : (X, τ) → (Y, σ) be a bijective and Dβ-continuous and Y is T0 space, then X is Dβ - T0.

Proof. Let x1, x2 be any two distinct elements of X. Since f is bijective map, then there exists y1, y2 ∈ Y with y1 ≠ y2 such that y1 = f(x1) and y2 = f(x2). Hence x1 = f-1(y1) and x2 = f-1(y2). Since Y is T0 space, therefore there exists an open set G in Y such that y1 ∈ G but y2 ∉ G. Since f is Dβ-continuous, f-1(G) is Dβ-open in X. Therefore y1 ∈ G implies f-1(y1) ∈ f-1(G) and y2 ∉ G implies f-1(y2) ∈ f-1(G) consequently x1 ∈ f-1(G) but x2 ∉ f-1(G). Therefore for any two distinct points y1, y2 in Y with y1 ∈ G and y2 ∉ G and G is open in Y, then there exists a Dβ-open set f-1(G) in X such that x1 ∈ f-1(G) but x2 ∉ f-1(G) . This shows that X is Dβ - T0.

Theorem 3.7. Let f : (X, τ) → (Y, σ) be a bijective and Dβ-open map and (X, τ) is T0 space, then (Y, σ) is Dβ - T0 space.

Proof. Let y1, y2 be any pair of distinct points of Y and f is one-one and onto, therefore there exists two distinct points x1, x2 in X such that f(x1) = y1 and f(x2) = y2. Since (X, σ) is T0 , therefore there exists an open set G in X such that x1 ∈ G but x2 ∉ G . Since f is Dβ-open map, then f(G) is Dβ-open in Y such that f(x1) = y1 ∈ f(G) but f(x2) = y2 ∉ f(G). Hence Y is Dβ - T0.

Theorem 3.8. Let f : (X, τ) → (Y, σ) be a bijective and Dβ-irresolute mapping and Y is Dβ - T0 space. Then X is also Dβ - T0.

Proof. Suppose x1, x2 be any two distinct points of X, then there exists y1, y2 ∈ Y with y1 ≠ y2 such that f(x1) = y1 and f(x2) = y2 . Since Y is Dβ - T0 , x1 = f-1(y1) and x2 = f-1(y2) . Since Y is Dβ-continuous and G is open in Y such that y1 ∈ G and y2 ∉ G. Since f is Dβ-irresolute, f-1(G) is Dβ-open in X. We have y1 ∈ G implies f-1(y1) = x1 ∈ f-1(G) but y2 ∉ G implies f-1(y2) = x2 ∉ f-1(G) . Therefore, x1, x2 ∈ X with x1 ≠ x2 , there exists a Dβ-open set f-1(G) in X such that x1 ∈ f-1(G) but x2 ∉ f-1(G) . Hence X is Dβ - T0.
Theorem 3.9. Let \( f : (X, \tau) \to (Y, \sigma) \) be bijective and \( D_\beta \)-continuous and \( Y \) is \( T_1 \) space, then \( X \) is \( D_\beta - T_1 \) space.

Proof. Let \( x_1, x_2 \) be any two distinct points of \( X \). By assumption there exists two points \( y_1, y_2 \in Y \) with \( y_1 \neq y_2 \) such that \( y_1 = f(x_1) \) and \( y_2 = f(x_2) \). Therefore \( x_1 = f^{-1}(y_1) \) and \( x_2 = f^{-1}(y_2) \). Since \( Y \) is \( T_1 \) space, there exists open sets \( M \) and \( N \) in \( Y \) such that \( y_1 = f(x_1) \in M \) but \( y_2 = f(x_2) \notin M \) and \( y_2 = f(x_2) \in N \) but \( y_1 = f(x_1) \notin N \). Since \( f \) is being \( D_\beta \)-continuous, \( f^{-1}(M) \) and \( f^{-1}(N) \) are \( D_\beta \)-open sets in \( X \) such that \( x_1 = f^{-1}(y_1) \in f^{-1}(M) \) but \( x_2 = f^{-1}(y_2) \notin f^{-1}(M) \) and \( x_2 = f^{-1}(y_2) \in f^{-1}(N) \) but \( x_1 = f^{-1}(y_1) \notin f^{-1}(N) \). This shows that \( X \) is \( D_\beta - T_1 \).

Theorem 3.10. \( f : (X, \tau) \to (Y, \sigma) \) be a bijective, \( D_\beta \)-open map and \( (X, \tau) \) is \( T_1 \) space, then \( (Y, \sigma) \) is \( D_\beta - T_1 \).

Proof. Let \( f : (X, \tau) \to (Y, \sigma) \) be a bijective, \( D_\beta \)-open map and \( X \) is \( T_1 \). Let \( y_1, y_2 \) be any pair of distinct points of \( Y \), then there exists two distinct points \( x_1, x_2 \in X \) such that \( f(x_1) = y_1 \) and \( f(x_2) = y_2 \). Since \( (X, \tau) \) is \( T_1 \), there exists open sets \( G \) and \( H \) in \( X \) such that \( x_1 \in G \) but \( x_2 \notin G \) and \( x_2 \in H \) but \( x_1 \notin H \). Since \( f \) is \( D_\beta \)-open map, \( f(G) \) and \( f(H) \) in \( Y \) such that \( f(x_1) = y_1 \in f(G) \) but \( f(x_2) = y_2 \notin f(G) \) and \( f(x_1) = y_1 \notin f(H) \) but \( f(x_2) = y_2 \in f(H) \). Therefore \( Y \) is \( D_\beta - T_1 \).

Remark: By referring the above mentioned results we can conclude that, a space being \( D_\beta - T_1 \) (for \( i = 0,1 \)) is a topological property.

Lemma 3.11. The set \( G \) is \( D_\beta \)-open in the space \( (X, \tau) \) if and only if for each \( x \in G \), there exists a \( D_\beta \)-open set \( F \) such that \( x \in F \subseteq G \).

Proof. Suppose \( G \) is any \( D_\beta \)-open set in \( (X, \tau) \), then for each \( x \in G \), let \( G = F \) such that \( x \in F \) and therefore \( x \in F \subseteq G \). Conversely, assume that for each \( x \in G \), there exists a \( D_\beta \)-open set \( F \) such that \( x \in F \subseteq G \). Therefore \( G = \bigcup \{ F_x : F_x \in D_\beta O(X) \} \), for each \( x \). Hence \( G \) is \( D_\beta \)-open in \( (X, \tau) \).

Theorem 3.12. A space \( (X, \tau) \) is \( D_\beta - T_1 \) space if and only if the singletons are \( D_\beta \)-closed sets.

Proof. Necessity- Let \( (X, \tau) \) be any \( D_\beta - T_1 \) space and suppose \( x \) be any point of \( X \). We show that \( \{ x \} \) is \( D_\beta \)-closed set. Suppose \( y \in \{ x \}^c \), then \( x \neq y \). Since \( X \) is \( D_\beta - T_1 \), therefore there exists two \( D_\beta \)-open sets \( G \) and \( H \) such that \( x \in G, y \in G \) and \( y \in H \), \( x \notin H \). This implies that \( y \in H \subseteq \{ x \}^c \), therefore by Lemma 3.11, the set \( \{ x \}^c \) is \( D_\beta \)-open, i.e. \( \{ x \} \) is \( D_\beta \)-closed.

Sufficiency- Let \( x, y \in X \) such that \( x \neq y \). It implies that \( \{ x \} \) and \( \{ y \} \) two disjoint \( D_\beta \)-closed sets in \( X \). Therefore \( \{ x \}^c, \{ y \}^c \) are two \( D_\beta \)-open sets such that \( y \notin \{ x \}^c \) but \( y \notin \{ y \}^c \) and \( x \notin \{ x \}^c \) but \( x \notin \{ y \}^c \). Hence \( X \) is \( D_\beta - T_1 \).

Theorem 3.13. The following statements are equivalent for a topological space \( (X, \tau) \);
(i) \( X \) is \( D_\beta - T_2 \).
(ii) Let \( x \in X \), for each \( y \in X \) with \( y \neq x \), there exists a \( D_\beta \)-open set \( G \) containing \( x \) such that \( y \notin C_{D_\beta}(G) \).
(iii) For each \( x \in X \), \( \bigcap \{ C_{D_\beta}(G) : G \) is \( D_\beta \)-open in \( X \) and \( x \in G \} = \{ x \} \)

Proof. (i) \( \Rightarrow \) (ii): Let \( X \) be any \( D_\beta - T_2 \) space and let \( x, y \in X \), then for each \( x \neq y \), there exists two disjoint \( D_\beta \)-open sets \( G \) and \( H \) such that \( x \in G, y \notin H \) and \( y \in H \), \( y \notin G \). Since \( H \) is \( D_\beta \)-open, therefore \( X \setminus H \) is \( D_\beta \)-closed and \( G \subseteq (X \setminus H) \). Thus we have, \( C_{D_\beta}(G) \subseteq C_{D_\beta}(X \setminus H) = (X \setminus H) \). Since \( y \notin (X \setminus H) \), therefore \( y \notin C_{D_\beta}(G) \).

(ii) \( \Rightarrow \) (iii): Let \( x \in X \), then for each \( y \neq x \) in \( X \), there exists a \( D_\beta \)-open set \( G \) such that \( x \in G \) and \( y \notin C_{D_\beta}(G) \). Therefore \( y \notin \bigcap \{ C_{D_\beta}(G) : G \) is \( D_\beta \)-open in \( X \) and \( x \in G \} = \{ x \} \).

(iii) \( \Rightarrow \) (i): Let \( x, y \in X \) such that \( x \neq y \), then by assumption \( y \notin \bigcap \{ C_{D_\beta}(G) : G \) is \( D_\beta \)-open in \( X \) and \( x \in G \} = \{ x \} \). This implies that \( y \notin C_{D_\beta}(G) \) where \( G \in D_\beta O(X) \) and \( x \in G \). This shows that \( y \in (X \setminus C_{D_\beta}(G)) \). Since \( C_{D_\beta}(G) \) is \( D_\beta \)-closed, then \( X \setminus C_{D_\beta}(G) = H \) (say) is \( D_\beta \)-open in \( X \) and \( y \in H \).

Therefore \( G \cap H = \emptyset \) and \( (X \setminus C_{D_\beta}(G)) = \emptyset \). Hence \( X \) is \( D_\beta - T_2 \).
4. **$D_\beta$-CLOSED GRAPH AND STRONGLY $D_\beta$-CLOSED GRAPH**

In this section we introduce $D_\beta$-closed graph and strongly $D_\beta$-closed graph and investigate some of their basic properties.

**Definition 4.1.** A function $f : (X, \tau) \to (Y, \sigma)$ has $D_\beta$-closed graph if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in D_\beta O(X, x)$ and $V \in \beta O(Y, y)$ such that $(U \times C_{\ell_\beta}(V)) \setminus G(f) = \phi$.

**Theorem 4.2.** A closed graph is always a $D_\beta$-closed graph.

**Proof.** The proof follows from the fact that every open set is $\beta$-open and $D_\beta$-open.

**Remark.** The converse of above theorem is not true in general.

**Example 4.3.** Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a, b, c, d\}, \{b\}\}$, then $(X, \tau)$ be a topological space. $C(X) = \{X, \emptyset, \{a, b, c, d\}, \{a\}, \{a, c, d\}\}$. Let $Y = \{1, 2, 3, 4\}$, $\sigma = \{Y, \emptyset, \{1, 2\}, \{1, 2, 4\}\}$, then $(Y, \sigma)$ be a topological space. $C(Y) = \{Y, \emptyset, \{3, 4\}, \{3\}\}$.

**Definition 4.4.** Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined by $f(a) = 2$, $f(b) = 3$, $f(c) = 4$ and $f(d) = 1$. Let $(c, d) \in D_\beta O(X)$ and $\{2\} \in \beta O(Y)$, then

$$
\left\{(c, d) \times \{2\}\right\} \cap \left\{(c, 4), \{d, 1\}\right\} =
\left\{(c, 2), \{d, 2\}\right\} \cap \left\{(c, 4), \{d, 1\}\right\} = \emptyset
$$

Therefore the graph $G(f)$ is $D_\beta$-closed graph, but it is not closed graph, since the $(c, d)$ is not open in $X$.

**Theorem 4.4.** The function $f : (X, \tau) \to (Y, \sigma)$ has a $D_\beta$-closed graph if and only if for each $(x, y) \in X \times Y$ such that $f(x) \neq y$, then there exits a $D_\beta$-open set $U$ and a $\beta$-open set $V$ containing $x$ and $y$, respectively such that $f(U) \cap C_{\ell_\beta}(V) = \phi$.

**Proof.** Suppose for each $(x, y) \in X \times Y$ with $f(x) \neq y$. Since $G(f)$ is $D_\beta$-closed graph, then there exits a $D_\beta$-open set $U$ and a $\beta$-open set $V$ containing $x$ and $y$, respectively such that $(U \times C_{\ell_\beta}(V)) \cap G(f) = \phi$.

Define $f : (X, \tau) \to (Y, \sigma)$ as a $D_\beta$-closed graph if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in D_\beta O(X, x)$ and $V \in \beta O(Y, y)$ such that $(U \times C_{\ell_\beta}(V)) \setminus G(f) = \phi$.

Examples of $D_\beta$-closed graph are given in Section 4.4.

**Sufficiency** - Let $(x, y) \not\in G(f)$, then $f(x) \neq y$ and therefore there exist a $\beta$-open set $U$ and an $\beta$-open set $V$ containing $x$ and $y$, respectively such that $f(U) \cap C_{\ell_\beta}(V) = \phi$. This implies for each $x \in U$ and for each $y \in C_{\ell_\beta}(V)$, $f(x) \neq y$. This proves that $(U \times C_{\ell_\beta}(V)) \cap G(f) = \phi$. Hence $f$ has a $D_\beta$-closed graph.

**Theorem 4.5.** If $f : (X, \tau) \to (Y, \sigma)$ is a $D_\beta$-continuous function from a space $X$ into a Hausdorff space $Y$, then $f$ has a $D_\beta$-closed graph.

**Proof.** Let $(x, y) \not\in G(f)$, then $f(x) \neq y$. Since $Y$ is Hausdorff space, there exist two disjoint open sets $P$ and $Q$ such that $f(x) \in Q$ and $y \in P$. Since $f$ is $D_\beta$-continuous, therefore by Theorem 3.9 of [16], there exists a $\beta$-open set $U$ in $X$ such that $x \in U$ and $f(U) \subseteq Q$. Therefore $f(U) \subseteq Y \setminus C_{\ell_\beta}(P)$. This implies that $f(U) \cap C_{\ell_\beta}(P) = \emptyset$. Since every open set is $\beta$-open, therefore $C_{\ell_\beta}(P) \subseteq C_{\ell_\beta}(P)$. Hence we get $f(U) \cap C_{\ell_\beta}(V) = \emptyset$, which implies that $f$ has a $D_\beta$-closed graph.

**Corollary 4.6.** If the function $f : (X, \tau) \to (Y, \sigma)$ has a $D_\beta$-closed graph, then for each $(x, y) \in X \times Y$ such that $f(x) \neq y$, then there exits two $D_\beta$-open sets $U$ and $V$ containing $x$ and $y$, respectively such that $(U \times C_{\ell_\beta}(V)) \cap G(f) = \emptyset$.

**Proof.** Suppose $f$ has a $D_\beta$-closed graph, then for each $(x, y) \in X \times Y$ such that $f(x) \neq y$, then there exits $\beta$-open sets $U$ and $\beta$-open set $V$ containing $x$ and $y$, respectively such that $(U \times C_{\ell_\beta}(V)) \cap G(f) = \emptyset$. Since every $\beta$-open set is $D_\beta$-open and therefore $C_{\ell_\beta}(V) \subseteq C_{\ell_\beta}(V)$, we have $(U \times C_{\ell_\beta}(V)) \cap G(f) = \emptyset$.

**Corollary 4.7.** The function $f : (X, \tau) \to (Y, \sigma)$ has a $D_\beta$-closed graph, then for each $(x, y) \in X \times Y$ such that $f(x) \neq y$, there exits two $D_\beta$-open sets $U$ and $V$ containing $x$ and $y$, respectively such that $f(U) \cap C_{\ell_\beta}(V) = \emptyset$.

**Proof.** Suppose the function $f : (X, \tau) \to (Y, \sigma)$ has a $D_\beta$-closed graph. By Theorem 4.4 for each $(x, y) \in X \times Y$ such that $f(x) \neq y$, there exits a $D_\beta$-open sets $U$ and a $\beta$-open set $V$ containing $x$ and $y$, respectively such that $f(U) \cap C_{\ell_\beta}(V) = \emptyset$. This implies for each $x \in U$ and for each $y \in C_{\ell_\beta}(V)$, $f(x) \neq y$. This proves that $(U \times C_{\ell_\beta}(V)) \cap G(f) = \emptyset$. Hence $f$ has a $D_\beta$-closed graph.
respectively such that \( f(U) \cap C_{\beta}(V) = \phi \). Since every \( \beta \) -closed set is \( D_\beta \) -closed, i.e., \( C_{\beta}(V) \subseteq C_{\beta}(V) \). Hence we get
\[
\text{f(U) \cap C_{\beta}(V) = \phi}.
\]

**Theorem 4.8.** If \( f : (X, \tau) \to (Y, \sigma) \) is a surjective function and has a \( D_\beta \) -closed graph from a space \( X \) onto a space \( Y \), then \( Y = \beta - T_2 \).

**Proof.** Let \( y_1 \) and \( y_2 \) be any two distinct points in \( Y \). Since \( f \) is surjective, then there exists a point \( x_1 \in X \) such that \( f(x_1) = y_1 \neq y_2 \). Thus \( (x_1, y_2) \not\in G(f) \). Since \( f \) has a \( D_\beta \)-closed graph, then by Theorem 4.4, there exists a \( D_\beta \)-open set \( U \) and a \( \beta \)-open set \( V \) containing \( x_1 \) and \( y_2 \), respectively such that \( f(U) \cap C_{\beta}(V) = \phi \). \( x_1 \in U \) implies \( f(x_1) = y_1 \in f(U) \). Thus \( x_1 \not\in C_{\beta}(V) \), then there exists a \( \beta \)-open set \( Y \setminus C_{\beta}(V) \) such that \( f(x_1) = y_1 \in Y \setminus C_{\beta}(V) \). Thus \( V \cap Y \setminus C_{\beta}(V) = \phi \). Hence \( Y = \beta - T_2 \).

**Corollary 4.9.** If \( f : (X, \tau) \to (Y, \sigma) \) is a surjective function and has a \( D_\beta \) -closed graph from a space \( X \) onto a space \( Y \), then \( Y = \beta - T_2 \).

**Proof.** It follows from the Theorem 4.5 and the Corollary 4.7.

**Definition 4.10.** A function \( f : (X, \tau) \to (Y, \sigma) \) has strongly \( D_\beta \) -closed graph if for each \( (x, y) \in (X \times Y) \setminus G(f) \), there exist \( U \in D_\beta \) \( O(X, x) \) and \( V \in \sigma(O(Y, y)) \) such that \( (U \times C_{\sigma}(V)) \setminus G(f) = \phi \).

**Remark.** Every strongly \( D_a \)-closed graph is strongly \( D_\beta \)-closed. But the converse is not true. Since it is shown in the Example 3.4 that the set \( A = \{a, d\} \) is \( D_\beta \)-closed but it is not \( D_a \)-closed.

**Theorem 4.11.** The function \( f : (X, \tau) \to (Y, \sigma) \) has strongly \( D_\beta \) -closed graph if and only if for each \( (x, y) \in (X \times Y) \) such that \( f(x) \neq y \), then there exists a \( D_\beta \)-open set \( U \) and an open set \( V \) containing \( x \) and \( y \), respectively such that \( f(U) \cap C_{\beta}(V) = \phi \).

**Proof.** Its proof is similar to the Theorem 4.4.

**Corollary 4.12.** If the function \( f : (X, \tau) \to (Y, \sigma) \) has a strongly \( D_\beta \) -closed graph, then for each \( (x, y) \in X \times Y \) such that \( f(x) \neq y \), then there exists a \( D_\beta \)-open set \( U \) and a \( \beta \)-open set \( V \) containing \( x \) and \( y \), respectively such that \( (U \times C_{\beta}(V)) \cap G(f) = \phi \).

**Proof.** Its proof is similar to the Corollary 4.6.


AUTHORS DETAILS

Purushottam Jha,,
Department of Mathematics,
Govt. P. G. College, Narayanpur, Chhattisgarh, India-494661
E-mail address: purush.jha@gmail.com

Manisha Shrivastava,,
Department of Mathematics,
Govt. J. Y. Chhattisgarh College,
Raipur, Chhattisgarh, India-492001
E-mail address: shrivastavanishtha9@gmail.com